46. *Stochastic Processes Built From Flows.**

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By virtue of the theory of semi-groups due to E. Hille[1] and the present author[2], we may construct stochastic processes in a separable measure space R from flows in R.

1. A flow in R is a one-parameter group $\{F_t\}$ of equi-measure transformations in R which is continuous in the sense that $f(F_t \cdot x)$, $f(x) \in L_p(R)$ $(1 \leqq p < \infty)$, is strongly continuous in t. Thus the flow $\{F_t\}$ induces a one-parameter group $\{T_t\}$ of linear operators in $L_p(R)$:

$$(T_t f)\ (x) = f(F_t \cdot x),\ f \in L_p(R), \tag{1.1}$$

$$T_t T_s = T_{t+s},\ T_0 = I \quad \text{(the identity)}, \tag{1.2}$$

$$\text{strong} \lim_{t \to t_0} T_t f = T_{t_0} f. \tag{1.3}$$

Each T_t is a transition operator in $L_p(R)$:

$$f(x) \geqq 0 \text{ implies } (T_t f)\ (x) \geqq 0 \text{ and } \int_R f(x)\, dx = \int_R (T_t f)\ (x)\, dx. \tag{1.4}$$

By the semi-group theory, $\{T_t\}$ admits infinitesimal generator A:

$$\begin{cases} Af = \text{strong} \lim\limits_{t \downarrow 0} \dfrac{T_t - I}{t} f \text{ for those } f \text{ which are dense in } L_p(R), \\ T_t f = \exp(tA) f = \text{strong} \lim\limits_{n \to \infty} \exp\left(nt[(I - n^{-1}A)^{-1} - I]\right) f, -\infty < t < \infty. \end{cases} \tag{1.5}$$

Since $(I - n^{-1}A)^{-1}$ exists as a transition operator for $n \gtrless 0$[3],

$$(I - n^{-1}A^2)^{-1} = (I - \sqrt{n^{-1}}A)^{-1} (I + \sqrt{n^{-1}}A)^{-1} \tag{1.6}$$

exists as a transition operator. Hence A^2 is the infinitesimal generator of a one-parameter semi-group $\{S_t\}$ of transition operators:

$$S_t f = \exp(tA^2) f,\ 0 \leqq t < \infty. \tag{1.7}$$

Thus the Fokker-Planck's equation in a Riemannian space R:

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* The following result was, under somewhat more restricted conditions and without proof, reported in May 1950 to the Conference in Probability of the International Congress of Mathematicians. Similar result with an interesting formulation was also obtained by Dr. Kiyosi Itô, by virtue of his theory of stochastic differential equations. See his paper in the same issue of this Proceedings.

1) Functional Analysis and Semi-groups, New York (1948).

2) On the differentiability and the representation of one-parameter semi-group of linear operators, J. Math. Soc. Japan, **1** (1948).

3) Since $\{T_t\}$ is a group (not only a semi-group).