70. *A Proof of the Hahn-Birkhoff Theorem. Notes on Banach Space (X).*

By Masahiro NAKAMURA.
Osaka Normal College, Tennôji, Osaka.
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In the preceding paper [2], it is pointed out that the Hahn-Birkhoff decomposition theorem on a strictly monotone Banach lattice can be extended to a complete Banach lattice having an order-continuous linear functional and can be proved basing on the idea of G. Birkhoff [1; 119] with a few modification. In this note we will give an alternative simpler proof of the theorem using Zorn's Lemma. In the sequel, to save the space, we use the notations and terminologies of G. Birkhoff's text book [1] without any explanations.

Let E be a complete Banach lattice having an *order-continuous* linear functional f, that is, $f(x_\alpha)$ converges to zero for any $x_\alpha \downarrow 0$, where we mean $x_\alpha \downarrow 0$ if $\{x_\alpha\}$ is a Moore-Smith set of E with $x_\alpha \leqq x_\beta$ for $\alpha > \beta$ and the infimum of x_α's is zero. After G. Birkhoff, the *positive, negative* and *null ideals* are defined as follows:

$$P = \{x \mid 0 < y \leqq |x| \rightarrow f(y) > 0\},$$
$$M = \{x \mid 0 < y \leqq |x| \rightarrow f(y) < 0\},$$
$$N = \{x \mid 0 < y \leqq |x| \rightarrow f(y) = 0\}.$$

These ideals are linearly independent normal subspaces as proved by G. Birkhoff. But by the sake of the completeness we will give it in full. Suppose that x and y belong to P. Then $0 < z \leqq |x+y|$ implies $0 < z \leqq |x| + |y|$, whence $z = x' + y'$ where $0 \leqq x' \leqq |x|$, $0 \leqq y' \leqq |y|$ and either x' or y' is strictly positive, and so $f(z) = f(x') + f(y')$ is strictly positive. That is, P is a subspace. Moreover, if x belongs to P and $|y| \leqq |x|$ then $0 < z \leqq |y|$ implies $0 < z \leqq |x|$, and so $f(z)$ is strictly positive, whence y belongs to P, that is, P is a normal subspace. Similarly, we may prove that M and N are also normal subspaces. To prove the linear independence, it is sufficient to show $P \frown M = M \frown N = N \frown P = 0$, or equivalently, each positive element belongs to at most one of such ideals. But this is plain, since x belongs to P if and only if $f(x)$ is strictly positive. This proves the above statement. (It is to be noted, that in the above the order-continuity of the functional is not used essentially, whence it is true for any linear functional).

Now we will use the order-continuity. Then *the above three ideals become complemented normal subspaces* in the sense of G. Birkhoff [1; 111]. By a remark of G. Birkhoff [1; 121], to prove