1. *On Riemannian Spaces Admitting a Family of Totally Umbilical Hypersurfaces. I.*

By Tyuzi ADATI.

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§1. Let V_n be an n-dimensional Riemannian space with the fundamental tensor $g_{\lambda\mu}$ $(\lambda,\mu,\nu,\ldots = 1, 2,\ldots, n)$ and assume that there exists a family of totally umbilical hypersurfaces

$$\sigma(x^\lambda) = \text{const.}. \tag{1.1}$$

If we denote the parametric representation of its hypersurfaces by $\quad x^\lambda = x^\lambda(x^i)\ (i,j,k,\ldots = \dot{1},\dot{2},\ldots,\dot{n}-\dot{1}),$
then from (1.1) we have by differentiation with respect to x^i

$$\sigma_\lambda B_i^{\cdot\lambda} = 0,$$

where $\quad \sigma_\lambda = \dfrac{\partial\sigma}{\partial x^\lambda},\ B_i^{\cdot\lambda} = \dfrac{\partial x^\lambda}{\partial x^i}.\quad$ Furthermore, differentiating with respect to x^j, we have

$$\sigma_{\lambda;\mu} B_i^{\cdot\lambda} B_j^{\cdot\mu} + \sigma_\lambda H_{ij}^{\cdot\cdot\lambda} = 0,$$

where $H_{ij}^{\cdot\cdot\lambda}$ is an Euler-Schouten's curvature tensor. If we denote the fundamental tensor and normals of the hypersurfaces by g_{ij} and B^λ respectively, we have, because of $H_{ij}^{\cdot\cdot\lambda} = Hg_{ij}B^\lambda$,

$$\sigma_{\lambda;\mu} B_i^{\cdot\lambda} B_j^{\cdot\mu} + H\sigma_\lambda B^\lambda g_{ij} = 0,$$

from which follows

$$(\sigma_{\lambda;\mu} + H\sigma_\nu B^\nu g_{\lambda\mu}) B_j^{\cdot\lambda} B_j^{\cdot\mu} = 0.$$

Consequently $\sigma_{\lambda;\mu}$ must take the form

$$\sigma_{\lambda;\mu} = \rho\, g_{\lambda\mu} + v_\lambda \sigma_\mu + v_\mu \sigma_\lambda\,, \tag{1.2}$$

where $\rho = -H\sigma_\nu B^\nu$ and v_λ is a certain vector.

Conversely, if (1.2) holds, we know easily that the hypersurfaces $\sigma(x^\lambda) = \text{const.}$ are totally umbilical.

Differentiating (1.2) and substituting the resulted equations in Ricci identities $\sigma_{\lambda;\mu\nu} - \sigma_{\lambda;\nu\mu} = -\sigma_\omega R^{\omega}_{\cdot\lambda\mu\nu}$, we have

$$\begin{aligned}-\sigma_\omega R^{\omega}_{\cdot\lambda\mu\nu} = &\{(\rho_\nu - \rho v_\nu)\, g_{\lambda\mu} - (\rho_\mu - \rho v_\mu)\, g_{\lambda\nu}\} \\ &+ \{(v_{\lambda;\nu} - v_\lambda v_\nu)\, \sigma_\mu - (v_{\lambda;\mu} - v_\lambda v_\mu)\, \sigma_\nu\} + \sigma_\lambda (v_{\mu;\nu} - v_{\nu;\mu})\,.\end{aligned} \tag{1.3}$$

If we put $\sigma_\lambda = \sqrt{\sigma^\mu\sigma_\mu}\, B_\lambda$, where $\sigma^\mu\sigma_\mu = g^{\mu\nu}\sigma_\mu\sigma_\nu$ and $B_\lambda = g_{\lambda\nu}B^\nu$, we have from (1.3)