12. *On Riemannian Spaces Admitting a Family of Totally Umbilical Hypersurfaces. II.*

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(Comm. by Z. SUETUNA, M.J.A., Feb. 12, 1951.)

§ 4. When orthogonal trajectories of the hypersurfaces $\sigma(x^\lambda)=$ const. are geodesics, (1.2) reduces to the form

$$\sigma_{\lambda;\,\mu} = \rho g_{\lambda\mu} + \eta\sigma_\lambda\sigma_\mu\,, \tag{4.1}$$

namely σ_λ is a torse-forming vector field. In this case, since v_λ is proportional to σ_λ and consequently $v_i = v_\lambda B_i^{\cdot\lambda} = 0$, (1.6) becomes

$$R_{jk} = B_j^{\cdot\mu} B_k^{\cdot\nu} R_{\mu\nu} + \beta g_{jk}\,.$$

Thus we have

Theorem 4.1. When orthogonal trajectories of the totally umbilical hypersurfaces $\sigma(x^\lambda) =$ const. are geodesics, in order that the hypersurfaces $\sigma(x^\lambda) =$ const. are Einstein spaces, it is necessary and sufficient that the tensor $\Pi_{\lambda\mu}$ takes the form

$$\Pi_{\lambda\mu} = ug_{\lambda\mu} + \zeta_\lambda\sigma_\mu + \zeta_\mu\sigma_\lambda\,.$$

Cor. 1. If σ_λ is a torse-forming vector field and $\Pi_{\lambda\mu} = ug_{\lambda\mu} + \kappa\sigma_\lambda\sigma_\mu$, then the hypersurfaces $\sigma(x^\lambda) =$ const. are Einstein spaces.

Cor. 2. If an Einstein space admits a torse-forming vector field σ_λ, then the hypersurfaces $\sigma(x^\lambda) =$ const. are also Einstein spaces.

We consider next a conformally flat space admitting a torse-forming vector field.

Differentiating (4.1) and substituting the resulted equations in Ricci identities $\sigma_{\lambda;\,\mu\nu} - \sigma_{\lambda;\,\nu\mu} = -\sigma_\omega R^{\omega}_{\cdot\cdot\lambda\mu\nu}$, we have

$$-\sigma_\omega R^{\omega}_{\cdot\cdot\lambda\mu\nu} = (\rho_\nu - \rho\eta\sigma_\nu)g_{\lambda\mu} - (\rho_\mu - \rho\eta\sigma_\mu)g_{\lambda\nu} + \sigma_\lambda(\eta_\nu\sigma_\mu - \eta_\mu\sigma_\nu)\,. \tag{4.2}$$

Multiplying by σ^λ and summing for λ, we have

$$(\rho_\nu + \sigma^\lambda\sigma_\lambda\eta_\nu)\sigma_\mu - (\rho_\mu + \sigma^\lambda\sigma_\lambda\eta_\mu)\sigma_\nu = 0\,,$$

from which follows that $\rho_\nu + \sigma^\lambda\sigma_\lambda\eta_\nu$ is proportional to σ_ν, that is, $\sigma^\lambda\sigma_\lambda\eta_\nu = a\sigma_\nu - \rho_\nu$, where a is a certain scalar. On the other hand, multiplying (4.2) by $g^{\lambda\mu}$ and summing for λ and μ, we have

$$\begin{aligned} -\sigma_\omega R^{\omega}_{\cdot\cdot\nu} &= (n-1)(\rho_\nu - \rho\eta\sigma_\nu) + \sigma^\lambda\sigma_\lambda\eta_\nu - \sigma^\lambda\eta_\lambda\sigma_\nu \\ &= (n-2)\rho_\nu + \{a - (n-1)\rho\eta - \sigma^\lambda\eta_\lambda\}\sigma_\nu\,. \end{aligned}$$

Thus we obtain the equations of the form