37. On the Simple Extension of a Space with Respect to a Uniformity. III.

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In the present note we discuss the completion of a space with respect to a uniformity. We make use of the same terminologies and notations as in the previous notes.[1]

§ 1. **The completion for the general case.** Let $\{\mathfrak{U}_\alpha; \alpha \in \Omega\}$ be a uniformity of a space R. Then the simple extension R^* of R with respect to $\{\mathfrak{U}_\alpha\}$ is complete with respect to the uniformity $\{\mathfrak{U}_\alpha^*\}$, in case $\{\mathfrak{U}_\alpha\}$ is a regular uniformity agreeing with the topology of R.[2] In the general case the simple extension R^*, however, is not always complete with respect to $\{\mathfrak{U}_\alpha^*\}$. We shall treat such a case in the following lines. In this case we construct the simple extension R^{**} of R^* with respect to the uniformity $\{\mathfrak{U}_\alpha^*\}$. Here we shall remark

Lemma 1. *The set of G^{**} for all open sets G of R is a basis of open sets of R^{**}.*

In case R^{**} is not complete we construct further the simple extension of R^{**}, and so on. We carry out our construction by transfinite induction. For the sake of convenience we write $R^{(0)}$, $R^{(1)}$, $R^{(2)}$, ... instead of R, R^*, R^{**}, ... Suppose that $R^{(\nu)}$ (and $G^{(\nu)}$ for open sets G of R) are defined for all ordinals ν less than an ordinal λ, and that $R^{(\nu)}$ are not complete, but with the following properties:

(1) For $0 \leqq \mu < \nu$ we have $R^{(\mu)} \subset R^{(\nu)}$ and $G^{(\nu)} \cdot R^{(\mu)} = G^{(\mu)}$.

(2) $G \subset H$ or $G \cdot H = 0$ implies $G^{(\nu)} \subset H^{(\nu)}$ or $G^{(\nu)} \cdot H^{(\nu)} = 0$.

(3) $\{G^{(\nu)}; G \text{ open in } R\}$ is a basis of open sets of $R^{(\nu)}$.

(4) Each point of $R^{(\nu)} - R$ is closed in $R^{(\nu)}$.

(5) $\mathfrak{U}_\alpha^{(\nu)} = \{U^{(\nu)}; U \in \mathfrak{U}_\alpha\}$ is an open covering of $R^{(\nu)}$.

(6) $\{S(x, \mathfrak{U}_\alpha^{(\nu)}); \alpha \in \Omega\}$ is a basis of neighbourhoods of each point x of $R^{(\nu)} - R$.

Here G, H are open sets of R.

In case λ is not a limit-number, we define $R^{(\lambda)}$ as the simple extension of $R^{(\lambda-1)}$ with respect to the uniformity $\{\mathfrak{U}_\alpha^{(\lambda-1)}; \alpha \in \Omega\}$. Then it is easily seen that $R^{(\lambda)}$ satisfies the conditions (1), (2), (3), (5), (6) for $\nu = \lambda$. If x is a point of $R^{(\lambda)} - R^{(\lambda-1)}$, then x is clearly a closed set of $R^{(\lambda)}$. Let $x \in R^{(\lambda-1)} - R$. Then we have $\bar{x} \cdot R^{(\lambda-1)} = x$.

1) K. Morita: On the simple extension of a space with respect to a uniformity. I, II. these Proc. **27**, No. 1, 2 (1951). These notes shall be cited with I., II. respectively.

2) Cf. I. § 5.