51. On the Metrization and the Completion of a Space with Respect to a Uniformity.

By Jingoro SUZUKI.

(Comm. by K. KUNUGI, M.J.A., May 16, 1951.)

We first recall some definitions.[1] A collection $\{\mathfrak{U}_\alpha \mid \alpha \in \Omega\}$ of open coverings of a topological space R is called a uniformity. If $\{\mathfrak{U}_\alpha \mid \alpha \in \Omega\}$ satisfies the condition:

For any $\alpha, \beta \in \Omega$ there exists $\gamma \in \Omega$ such that $\mathfrak{U}_\gamma$ is a refinement of $\mathfrak{U}_\alpha$ and $\mathfrak{U}_\beta$, $\{\mathfrak{U}_\alpha\}$ is called a T-uniformity.

If $\{\mathfrak{U}_\alpha \mid \alpha \in \Omega\}$ satisfies the condition:

For any $\alpha \in \Omega$ there exists $\lambda(\alpha) \in \Omega$ such that for each set $U_\lambda(\alpha) \in \mathfrak{U}_\lambda(\alpha)$ we can determine a set U_α of $\mathfrak{U}_\alpha$ and $\delta = \delta(\alpha, U_{\lambda(\alpha)}) \in \Omega$ so that $S(U_{\lambda(\alpha)}, \mathfrak{U}_\delta) \subset U_\alpha$, the uniformity $\{\mathfrak{U}_\alpha\}$ is called regular.

In § 1 we shall prove

Theorem 1. If a countable number of open coverings $\{\mathfrak{U}_n \mid n = 1, 2, \cdots\}$ of a T_1-space R forms a regular T-uniformity agreeing with the topology, then R is metrizable.

The simple extension R^* of a space R with respect to a uniformity $\{\mathfrak{U}_\alpha\}$ is not always complete. In § 2 we shall show that if we understand the notion of a Cauchy family in a more restricted sense, then the simple extension R^+ of R in this restricted sense is complete if $\{\mathfrak{U}_\alpha\}$ agrees with the topology of R.

I express my sincere thanks to Prof. K. Morita for his many valuable suggestions and advices.

§ 1. Theorem 1 will be established by virtue of a theorem of A.H. Frink,[2] if the following three lemmas are proved.

Lemma 1. Under the assumption of the theorem there exists a uniformity $\{\mathfrak{V}_n \mid n = 1, 2, \cdots\}$ such that $\{\mathfrak{V}_n\}$ is equivalent to $\{\mathfrak{U}_n\}$ and $\mathfrak{V}_1 > \mathfrak{V}_2 > \cdots > \mathfrak{V}_n > \cdots$.

Proof. We put $\mathfrak{U}_1 = \mathfrak{V}_1$. Next we select $\mathfrak{U}_{\beta_2}$ such that $\mathfrak{U}_{\lambda(1)}, \mathfrak{U}_2 > \mathfrak{U}_{\beta_2}$ and put $\mathfrak{U}_{\beta_2} = \mathfrak{V}_2$. Now let us assume that $\mathfrak{V}_i$ are obtained for $i \leqq n$. We take $\mathfrak{U}_{\beta_{n+1}}$ such that $\mathfrak{U}_{\lambda(\beta_n)}, \mathfrak{U}_{n+1} > \mathfrak{U}_{\beta_{n+1}}$ and put $\mathfrak{U}_{\beta_{n+1}} = \mathfrak{V}_{n+1}$. Then $\{\mathfrak{V}_n \mid n = 1, 2, \cdots\}$ satisfies clearly the conditions of Lemma 1.

Lemma 2. For any point p of the space R and any index n, there exists an index m_0 such that

1) K. Morita: On the simple extension of a space with respect to a uniformity. I. Proc. Japan Acad. **27** No. 2, (1951).

2) A. H. Frink: Distance functions and the metrization problem. Bull. Amer. Math. Soc., vol. XLIII (1937), Theorem 4, p. 141.