

136. On the Metrizable Condition.¹⁾

By Masahiro SUGAWARA.

(Comm. by K. KUNUGI, M.J.A., Dec. 12, 1951.)

L. W. Cohen considered a neighborhood space S topologized by the neighborhood basis $\{U_\alpha(p)\}$ of each point p of S , where α is an element of a set A , such that:

$$\text{I. } \bigcap_{\alpha \in A} U_\alpha(p) = \{p\},$$

II. For each $\alpha, \beta \in A$ and $p \in S$ there exists a $\gamma = \gamma(\alpha, \beta; p) \in A$ such that $U_\gamma(p) \subset U_\alpha(p) \cap U_\beta(p)$,

III. For each $\alpha \in A$ and $p \in S$ there exist $\lambda(\alpha), \delta(p, \alpha) \in A$ such that, if $U_{\delta(p, \alpha)}(q) \cap U_{\lambda(\alpha)}(p) \neq 0$, then $U_{\delta(p, \alpha)}(q) \subset U_\alpha(p)$;

and he showed that such a space S can be imbedded in a complete space S^* ²⁾. He gave also a question as whether a Hausdorff space satisfying the first denumerability axiom and condition III is metrizable. This note is concerned with this question.

We consider the next condition IV instead of III,

IV. For each $\alpha \in A$ and $p \in S$ there exist $\lambda'(p, \alpha), \delta'(p, \alpha) \in A$ such that, if $U_{\delta'(p, \alpha)}(q) \cap U_{\lambda'(p, \alpha)}(p) \neq 0$, then $U_{\delta'(p, \alpha)}(q) \subset U_\alpha(p)$.

Our result is

Theorem. *If the neighborhood space S satisfies the first denumerability axiom, i.e., the set of suffix of neighborhood basis can be taken to the set N of integers, and the above condition I, II and IV, then S is metrizable.*

As a Hausdorff space satisfies the condition I and II, and the condition III implies IV, this theorem gives the affirmative answer to the question.

To prove the theorem, we use

Frink's Theorem³⁾. *A necessary and sufficient condition that a neighborhood space S be metrizable is that for every point $p \in S$ there exists a sequence of neighborhoods $\{U_n(p)\}$, monotone decreasing and whose logical product is $\{p\}$, selected from the original neighborhoods and equivalent to them, satisfying the condition:*

V. *For each $n \in N$ and $p \in S$ there exists $m = m(p, n) \in N$ such that $m \geq n$ and if $U_m(q) \cap U_n(p) \neq 0$ then $U_m(q) \subset U_n(p)$.*

Let S satisfy the assumption of the theorem, i.e., for each point

1) This was reported at the annual meeting of Math. Soc. of Japan in Oct. 1950, and has been completed by the encouragement of Prof. T. Inagaki.

2) L. W. Cohen: On imbedding a space in a complete space, Duke Math. Jour. vol. 5(1935), p. 183.

3) A. H. Frink: Distance function and the metrization problem, Bull. Amer. Math. Soc. vol. 43 (1937), Theorem 4 p. 141.