7. On Some Representation Theorems in an Operator Algebra. III.

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When we discuss group algebras on a locally compact group, the notion of C*-algebra is very useful. But some group algebras are not always C*-algebra. Hence we shall introduce a notion of normed algebra including C*- and group algebras.

5. A normed*-algebra $\mathfrak{A}$ with the norm $\|\cdot\|$ over the complex number field is called with *D^*-algebra* if $\mathfrak{A}$ has an approximate identity $\{e_\alpha\}$ (cf. Segal [1]) which is a directed set such as $\|e_\alpha\| \leq 1$ and $\|e_\alpha x - x\| \underset{\alpha}{\rightarrow} 0$ for any $x \in \mathfrak{A}$. Clearly C*- and L^1-group algebras are D*-algebras. $f(\cdot)$ is said to be *semi-trace* of $\mathfrak{A}$ if it is a linear functional (not always bounded) defined on a dense subalgebra generated by $\{xy;\ x, y \in \mathfrak{A}\}$ such that $f(xy) = f(yx)$, $f(x^*) = \overline{f(x)}$ $f(x^*x) \geq 0$ and $f((xy)^*xy) \leq \|x\|^2 f(y^*y)$. A semi-trace is *pure*, if it is not a linear combination of two linearly independent semi-traces of $\mathfrak{A}$. A representation $\{U_x, \mathfrak{H}\}$ is said to be *proper*, if the element $\xi \in \mathfrak{H}$ satisfying $U_x \xi = 0$ for all $x \in \mathfrak{A}$ is only the zero element O in $\mathfrak{H}$. Moreover we also define the properness for a two-sided representation $\{U_x, V_x, j, \mathfrak{H}\}$, if the one-sided representation $\{U_x, \mathfrak{H}\}$ or $\{V_x, \mathfrak{H}\}$ is proper. Denote the W*-algebras generated by $\{U_x;\ x \in \mathfrak{A}\}$ or $\{V_x;\ x \in \mathfrak{A}\}$ by $\boldsymbol{U}$ or $\boldsymbol{V}$ respectively.

Theorem 5. *Let τ be a semi-trace of a D^*-algebra $\mathfrak{A}$. Then there corresponds a proper two-sided representation $\{U_x, V_x, j, \mathfrak{H}\}$ such that $\boldsymbol{U} = \boldsymbol{V}'$, $\boldsymbol{U}' = \boldsymbol{V}$ and $jAj = A^*$ for $A \in \boldsymbol{U} \frown \boldsymbol{V}$.*

Corollary 5.1. *If the D^*-algebra $\mathfrak{A}$ is separable, then the semi-trace is a directed integral of pure semi-traces $\pi(\cdot, \lambda)$, $\lambda \bar{\in} N$ ($\sigma(\lambda)$-null set), with respect to a $\sigma(\lambda)$-measure.*

A two-sided representation $\{U_x, V_x, j, \mathfrak{H}\}$ is *strictly normal*, if there exists $\xi \in \mathfrak{H}$ such that $U_x\xi = V_x\xi$ for all $x \in \mathfrak{A}$ and $\{U_x\xi;\ x \in \mathfrak{A}\}$ span $\mathfrak{H}$. Then

Theorem 6. *If a D^*-algebra $\mathfrak{A}$ has a strictly normal two-sided representation $\{U_x, V_x, j, \mathfrak{H}\}$, then the normalizing function is trace and conversly. This correspondence is one-to-one without equivalence. The generated W^*-algebra $\boldsymbol{U}$ (or $\boldsymbol{V}$) has a complete trace and both in finite class (in the sense of J. Dixmier [3]).*

6. Motion in C*- or D*-algebra. The investigation of a motion in C*-algebra has been introduced by I. E. Segal (cf. [2]) which is