5. *On Some Problems of Birkhoff.*

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In this paper we shall deal with problems 65 and 7 in Lattice Theory by G. Birkhoff[1] and discuss some axioms of lattices in connection with the latter problem.

§ 1. G. D. Birkhoff and G. Birkhoff[2] have developed a very brief set of postulates for distributive lattices.

Theorem. Any algebraic system which satisfies the following postulates for all a, b, c is a distributive lattice with I.

(1)	$a \frown a = a$		for all a
$(2)_1$	$a \smile I = I$	$(2)_2$	$I \smile a = I$
$(3)_1$	$a \frown I = a$	$(3)_2$	$I \frown a = a$ for some I and all a
$(4)_1$	$a \frown (b \smile c) = (a \frown b) \smile (a \frown c)$	$(4)_2$	$(b \smile c) \frown a = (b \frown a) \smile (c \frown a)$.

Problem 65 is to prove or disprove the independence of the seven identities assumed as postulates in the above theorem.

We shall show that these identities are independent of each other.

I. *Independence of (1)*

We consider a system of three sets $I = \{1, 2, 3\}$, $a = \{1, 2\}$, $b = \{1\}$. About their join and meet operations, we take set-theoretical sum and intersection, except the case: $a \frown a = b$.

For this system we can easily show that the six postulates (2)–(4) are satisfied but (1) is not. Concerning $(4)_1$ we have, for instance

x	y	z	$x \frown (y \smile z)$	$(x \frown y) \smile (x \frown z)$
a	a	a	$a \frown (a \smile a) = a \frown a = b$,	$(a \frown a) \smile (a \frown a) = b \smile b = b$
a	a	b	$a \frown (a \smile b) = a \frown a = b$,	$(a \frown a) \smile (a \frown b) = b \smile b = b$.

All other cases are treated similarly. The other identities except (1) are verified easily.

II. *Independence of (2)*

Let $\mathfrak{S}$ be a family of all subsets of a set $\mathfrak{A}$ ($\mathfrak{S}$ does not contain null set.) Define join and meet operations as follows,

$a \smile b = a$

$a \frown b = ab$ (where ab means set-theoretical intersection of sets a, b)

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1) G. Birkhoff: Lattice Theory, 1948.

2) G. D. Birkhoff and G. Birkhoff: Distributive postulates for systems like Boolean algebra. Trans. Am. Math. Soc. **60** (1946).