44. On The Interval Containing At Least One Prime Number.

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(Comm. by Z. SUETUNA, M.J.A., April 12, 1952.)

Bertrand-Tschebyschef's theorem (1852) is well-known for the interval between \( x \) and \( 2x \) where \( x > 1 \), within which at least one prime number exists; this paper, however, enables us to reduce it up to between \( x \) and \( 6x/5 \) where \( x \geq 25 \). In conformity with Ramanujan*, we establish the proof of our theorem upon the following fundamental formula: 

\[ T(x) = \sum_{m=1}^{\infty} \psi(x/m) = \log \Gamma([x]+1) \]

where \( \psi(x) = \sum_{p \leq x} \log p \).

Lemma 1. When \( n > 1 \),

\[ \frac{1}{n} T(x) - T\left(\frac{x}{n}\right) \geq \frac{1}{n} \log \Gamma(x) - \log \Gamma\left(\frac{x+n-1}{n}\right) \quad (x \geq 1) \]

and

\[ \frac{1}{n} T(x) - T\left(\frac{x}{n}\right) \leq \frac{1}{n} \log \Gamma(x+1) - \log \Gamma\left(\frac{x+1}{n}\right) \quad (x \geq n). \]

Proof. Since \( \frac{\Gamma''(s)}{\Gamma'(s)} = \int_0^\infty \left(\frac{e^{-t}}{1-e^{-t}}\right) dt \) when \( s > 0 \),

\[ \frac{\Gamma''(x)}{\Gamma'(x)} - \frac{\Gamma''(x+n-1)}{\Gamma'(x+n-1)} = \int_0^\infty \frac{1}{1-e^{-t}} \left( e^{-\frac{x+n-1}{n}t} - e^{-xt} \right) dt > 0 \quad (x > 1) \]

and

\[ \frac{\Gamma''(x+1)}{\Gamma'(x+1)} - \frac{\Gamma''(x+1+n)}{\Gamma'(x+1+n)} = \int_0^\infty \frac{1}{1-e^{-t}} \left( e^{-\frac{x+1}{n}t} - e^{-\frac{x+1+n}{n}t} \right) dt > 0 \quad (x > 0), \]

that is to say,

\[ \frac{1}{n} \log \Gamma(x) - \log \Gamma\left(\frac{x+n-1}{n}\right) \quad \text{and} \quad \frac{1}{n} \log \Gamma(x+1) - \log \Gamma\left(\frac{x+1}{n}\right) \]

are increasing functions when \( x \geq 1 \) and \( x > 0 \) resp.

Hence we have

\[ \frac{1}{n} \log \Gamma(x) - \log \Gamma\left(\frac{x+n-1}{n}\right) \]

\[ \leq \frac{1}{n} \log \Gamma([x]+1) - \log \Gamma\left(\frac{[x]+n}{n}\right) \quad (x \geq 1), \]

\[ \leq \frac{1}{n} \log \Gamma([x]+1) - \log \Gamma\left(\frac{x}{n}\right) + \frac{1}{n} T(x) - T\left(\frac{x}{n}\right) \]

\[ \leq \frac{1}{n} \log \Gamma([x]+1) - \log \Gamma\left(\frac{[x]+1}{n}\right) \quad ([x] \geq n-1), \]

\[ \leq \frac{1}{n} \log \Gamma(x+1) - \log \Gamma\left(\frac{x+1}{n}\right) \quad (x > 0); \]

* S. Ramanujan: A Proof of Bertrand's postulate (Collected papers, 208–209).