44. On The Interval Containing At Least One Prime Number.

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Bertrand-Tschebyschef's theorem (1852) is well-known for the interval *between* x *and* $2x$ *where* $x>1$, within which at least one prime number exists; this paper, however, enables us to reduce it up to *between* x *and* $6x/5$ *where* $x\geqq 25$. In conformity with *Ramanujan**, we establish the proof of our theorem upon the following fundamental formula: $T(x)=\sum_{m=1}^{\infty}\psi(x/m)=\log\Gamma([x]+1)$ where $\psi(x)=\sum_{m=1}^{\infty}\vartheta(\sqrt[m]{x})$ and $\vartheta(x)=\sum_{p\leqq x}\log p$.

Lemma 1. *When* $n>1$,

$$\frac{1}{n}T(x)-T\left(\frac{x}{n}\right)\geqq\frac{1}{n}\log\Gamma(x)-\log\Gamma\left(\frac{x+n-1}{n}\right)\qquad(x\geqq 1)$$

and $$\frac{1}{n}T(x)-T\left(\frac{x}{n}\right)\leqq\frac{1}{n}\log\Gamma(x+1)-\log\Gamma\left(\frac{x+1}{n}\right)\qquad(x\geqq n).$$

Proof. Since $\frac{\Gamma'}{\Gamma}(s)=\int_0^\infty\left(\frac{e^{-t}}{t}-\frac{e^{-st}}{1-e^{-t}}\right)dt$ when $s>0$,

$$\frac{\Gamma'}{\Gamma}(x)-\frac{\Gamma'}{\Gamma}\left(\frac{x+n-1}{n}\right)=\int_0^\infty\frac{1}{1-e^{-t}}\left(e^{-\frac{x+n-1}{n}t}-e^{-xt}\right)dt>0\quad(x>1)$$

and $$\frac{\Gamma'}{\Gamma}(x+1)-\frac{\Gamma'}{\Gamma}\left(\frac{x+1}{n}\right)=\int_0^\infty\frac{1}{1-e^{-t}}\left(e^{-\frac{x+1}{n}t}-e^{-(x+1)t}\right)dt>0\quad(x>0),$$

that is to say, $\frac{1}{n}\log\Gamma(x)-\log\Gamma\left(\frac{x+n-1}{n}\right)$ and $\frac{1}{n}\log\Gamma(x+1)-\log\Gamma\left(\frac{x+1}{n}\right)$ are increasing functions when $x\geqq 1$ and $x>0$ resp.

Hence we have

$$\begin{aligned}
&\frac{1}{n}\log\Gamma(x)-\log\Gamma\left(\frac{x+n-1}{n}\right)\\
&\leqq\frac{1}{n}\log\Gamma([x]+1)-\log\Gamma\left(\frac{[x]+n}{n}\right)&&(x\geqq 1),\\
&\leqq\frac{1}{n}\log\Gamma([x]+1)-\log\Gamma\left(\left[\frac{x}{n}\right]+1\right)=\frac{1}{n}T(x)-T\left(\frac{x}{n}\right)\\
&\leqq\frac{1}{n}\log\Gamma([x]+1)-\log\Gamma\left(\frac{[x]+1}{n}\right)&&([x]\geqq n-1),\\
&\leqq\frac{1}{n}\log\Gamma(x+1)-\log\Gamma\left(\frac{x+1}{n}\right)&&(x>0);
\end{aligned}$$

* S. Ramanujan: A Proof of Bertrand's postulate (Collected papers, 208-209).