78. On Functions Harmonic in a Circle, with Special Reference to Poisson Representation.

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1. We consider a family of functions harmonic in the unit circle of the $z=re^{i\theta}$-plane. It is well known that the Dirichlet problem, i. e. the first boundary value problem on harmonic functions, for the unit circle is solved by the *Poisson integral formula*

$$u_0(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|e^{i\varphi}-z|^2}f(\varphi)d\varphi,$$

in the sense that, $f(\varphi)$ being prescribed as any boundary value function integrable for $0\leqq\varphi<2\pi$, the function $u_0(z)$ defined by the formula is harmonic in $|z|<1$ and tends to $f(\varphi)$ almost everywhere in $0\leqq\varphi<2\pi$ as z tends to $e^{i\varphi}$ along a Stolz path.

The Poisson formula, especially in case of bounded boundary values, is characterized by its special behavior that, if $f(\varphi)$ is restricted by $f_s\leqq f(\varphi)\leqq f_a$ for $0\leqq\varphi<2\pi$, then the function $u_0(z)$ associated to $f(\varphi)$ by the formula submits to the same restriction $f_s\leqq u_0(z)\leqq f_a$ in $|z|<1$.

However, in case of unbounded boundary values, the circumstance becomes somewhat complicated. Although, for instance, a function

$$\frac{1-|z|^2}{|e^{i\varphi}-z|^2}\equiv\Re\frac{e^{i\varphi}+z}{e^{i\varphi}-z}$$

is harmonic in $|z|<1$ for any fixed φ and has boundary values vanishing everywhere except at $e^{i\varphi}$ alone, it must once be excluded to add a linear combination of such functions with various φ's or of unbounded functions of analogous character. The uniqueness of the solution of Dirichlet problem in the proper sense can thus be verified. The Poisson formula displays its effect concerning Dirichlet problem essentially in a range of the bounded harmonic functions, while it may be suitably generalized to a certain extent.

In the present Note, we shall discuss the problems on Poisson integral from the latter version, especially those characterizing the family of functions representable by Poisson integral.

2. We begin with an extremal property of Poisson integral for functions bounded in one side.

Theorem 1. *Let $f(\varphi)$ be a function integrable for $0\leqq\varphi<2\pi$ and*