11. Probability-theoretic Investigations on Inheritance. XVI$_4$. Further Discussions on Interchange of Infants

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8. Comparison between several probabilities

We state here some inequalities supplementing those mentioned in § 3. From their respective definitions, we get immediately the inequalities

(8.1) $$\Psi(ij)\leqq\Phi_*(ij),\qquad \Psi_*(ij,\ hk)\leqq\Phi(ij,\ hk),$$

and consequently

(8.2) $$F(ij)\leqq\mathfrak{F}(ij),\qquad \mathfrak{G}(ij,\ hk)\leqq G(ij,\ hk).$$

We thus conclude the inequalities $\Psi\leqq\Phi_*$, $\Psi_*\leqq\Phi$ and hence

(8.3) $$F\leqq\mathfrak{F}\equiv\mathfrak{G}\leqq G,$$

which are also evident by definitions. These inequalities can also be verified directly from their final expressions. For instance, by rearranging the terms in $\mathfrak{F}-F$, we get

$$\begin{aligned}\mathfrak{F}-F=&(S_2-S_3)+3(S_3-S_4)+3(S_2^2-S_4)-(S_4-S_5)-18(S_2S_3-S_5)\\&-(S_5-S_6)-9S_2(S_2^2-S_4)+17(S_2S_4-S_6)+8(S_3^2-S_6)+8S_3(S_2^2-S_4)\\&\qquad+4(S_3S_4-S_7)-12(S_2S_5-S_7)\\=&\sum_{i,j}{}'p_ip_j\{(p_i+p_j)+3(p_i^2+p_j^2)+6p_ip_j-(p_i^3+p_j^3)-18p_ip_j(p_i+p_j)\\&-(p_i^4+p_j^4)-18S_2p_ip_j+17p_ip_j(p_i^2+p_j^2)+16p_i^2p_j^2+16S_3p_ip_j\\&\qquad+4p_i^2p_j^2(p_i+p_j)-12p_ip_j(p_i^3+p_j^3)\}\\=&\sum_{i,j}{}'p_ip_j\{2(p_i^2+p_j^2)(p_i-p_j)^2+4p_ip_j(p_i+p_j)(p_i^2+p_j^2)\\&\qquad+18p_ip_j((1-p_i-p_j)^2-(S_2-p_i^2-p_j^2))\\&+16p_ip_j(S_3-p_i^3-p_j^3)+(p_i+p_j)(8(p_i^2+p_j^2)+p_ip_j)(1-p_i-p_j)\\&+6((p_i-p_j)^2+p_ip_j)(1-p_i-p_j)^2+(p_i+p_j)(1-p_i-p_j)^3\},\end{aligned}$$

the last member remaining evidently always non-negative, since

$$(1-p_i-p_j)^2-(S_2-p_i^2-p_j^2)=\sum_{h,k\neq i,j}{}'2p_hp_k\geqq0.$$

For the difference $G-\mathfrak{G}$, we get similarly

$$\begin{aligned}G-\mathfrak{G}=&2S_2(S_2-S_3)+2(S_2S_3-S_5)+2S_2(S_3-S_4)\\&\qquad+2S_2(S_2^2-S_4)-2(S_5-S_6)\\&-8(S_2S_4-S_6)-5(S_3^2-S_6)-16S_2(S_2S_3-S_5)\\&\qquad+12(S_2S_5-S_7)+16(S_3S_4-S_7)\\&-4S_2^2(S_2^2-S_4)+12S_2(S_2S_4-S_6)-5(S_4^2-S_8)-4(S_2S_6-S_8)\end{aligned}$$