23. On a Sufficient Condition for a Tensor to be Harmonic

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We prove the following theorem, which is an extension of Besicovitch's theorems[1].

Theorem. *Let α be a p-form*[2] *on an n-dimensional Riemannian space* C^∞,[2] *M, and $a_{i_1 i_2 \dots i_p}$ be its bounded coefficients which are defined and continuous on M except at most at the points of a set E_1 of $(n-1)$-dimensional measure 0 and further which is totally differentiable and satisfies*

$$d\alpha = \delta\alpha = 0$$

at every point except at most those of a set E_2 expressible as the sum of an enumerable infinity of sets of finite $(n-1)$-dimensional measure; then α is harmonic (in Hodge's sense) on M.

Lemma. *Suppose that F is a continuous additive function of an interval in the space R_n, such that $F(I)/[\delta(I)]^a$ is bounded and which fulfils the condition (1_a) at every point except at most those of a set E_1^* of measure (Λ_a) 0, where I is an arbitrary interval and $0 \leqslant a < n$, and that g is a summable function. Suppose further that* (i) *$(\mathfrak{Q})\underline{F}(x) > -\infty$ at every point x except at most those of a set E_2^* expressible as the sum of an enumerable infinity of sets of finite measure (Λ_a), and that* (ii) *$(\mathfrak{Q})\underline{F}(x) \geqslant g(x)$ at almost all points x; then*

$$F(I_0) \geqslant \int_{I_0} g(x)\, dx$$

for every interval I_0.

A particular case of this lemma is stated in Saks' "Theory of the integral", p. 193. This lemma can be proved quite similarly. Also for the notations and the terms used in it, see this book.

Proof of the theorem. We may clearly suppose that M is a domain D on R_n with the coordinate-system $x_1, x_2, \dots, x_n$,[3] and D contains the interval $I_1: 0 \leqq x_1 \leqq 1,\ 0 \leqq x_2 \leqq 1, \dots, 0 \leqq x_n \leqq 1$, and further

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1) A. S. Besicovitch: On sufficient conditions for a function to be analytic and on behaviour of analytic functions in the neighbourhood of non-isolated singular points, Proc. Lond. Math. Soc. (2) **32**, 1–9 (1931). The two theorems stated in this note can be brought to a combined theorem, which is a particular case of our theorem. Though this is not so difficult to see, perhaps none has remarked it. Besicovitch's original proofs are not applicable to this combined theorem.

2) See de Rham and Kodaira: Harmonic Integrals, Mimeographed Notes, Institute for Advanced Study, Princeton (1950).

3) The Riemannian metric defined on D is however, of course, not necessarily Euclidean.