37. On the Jordan-Hölder-Schreier Theorem

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In this note we shall formulate the Jordan-Hölder-Schreier Theorem for groups in any lattice. This formulation is the extension of the usual Jordan-Hölder-Schreier Theorem for modular lattices, and of the Jordan-Hölder Theorem for composition series of lower semi-modular lattices.

Let L be a lattice. In the following we denote the elements of L by small latters $a, b, x, y, m, n, \cdots$. By m/n we mean the closed interval $\{x;\ m\geqq x\geqq n,\ x\in L\}$, and by $m/$ the principal ideal generated by m in L.

Definition 1. An element $a\in m/n$ is called m/n-modular if and only if

1) $x, y\in m/n,\ x\geqq a$ implies $(x\frown y)\smile a=x\frown(y\smile a)$ and

2) $x, y\in m/n,\ x\geqq y$ implies $(x\frown a)\smile y=x\frown(a\smile y)$.

Remark. Putting $m/$-modular in place of m/n-modular in this definition, we can argue similarly in the following arguments.

Theorem 1. If $a, b\in m/n$ and a is m/n-modular, then the correspondences $x\to x\frown b$ and $y\to y\smile a$ are inverse isomorphisms between $a\smile b/a$ and $b/a\frown b$.

Proof. This theorem is immediate from the above definition.

Theorem 2. If a is m/n-modular and $b\in m/n$, then $a\frown b$ is b/n-modular.

Proof. (i) If $x, y\in b/n$ and $x\geqq a\frown b$ then

$$\begin{aligned}
&(x\frown y)\smile(a\frown b)\\
&=[(x\frown y)\smile a]\frown b\\
&=\{[(a\smile x)\frown b\frown y]\smile a\}\frown b \qquad \text{(applying Theorem 1)}\\
&=(a\smile x)\frown[(b\frown y)\smile a]\frown b\\
&\geqq x\frown(y\smile a)\frown b\\
&=x\frown[y\smile(a\frown b)]\\
&\geqq(x\frown y)\smile(a\frown b).
\end{aligned}$$

Hence we have

$$(x\frown y)\smile(a\frown b)=x\frown[y\smile(a\frown b)]$$

(ii) If $x, y\in b/n$ and $x\geqq y$, then we have

$$\begin{aligned}
&x\frown[(a\frown b)\smile y]\\
&=x\frown[b\frown(a\smile y)]\\
&=x\frown(a\smile y)\\
&=(x\frown a)\smile y\\
&=[x\frown(a\frown b)]\smile y.
\end{aligned}$$