96. On Selberg's Function

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1. In a recent paper, A. Selberg has achieved an elementary proof of Dirichlet's theorem about primes in an arithmetic progression[5] (numbers in square brackets refer to the references at the end of this note), and his proof is based upon the following Selberg's Inequality:

$$(1)\qquad \frac{x}{k}V(x)=\sum_{p\leq x,\ p\equiv\lambda(k)}\log^2 p+\sum_{pq\leq x,\ pq\equiv\lambda(k)}\log p\log q+O(x),$$

where

$$(2)\qquad V(x)=\sum_{d\leq x,\ (d,k)=1}\frac{\mu(d)}{d}\log^2\frac{x}{d}=\frac{2}{\varphi(k)}x\log x+O(x).$$

For every positive integer k, $\mu(k)$ and $\varphi(k)$ are the Möbius function and the Euler function respectively. p,q are primes and $(k,l)=1$.

We shall give in this note the generalized forms of (1) and (2) (Theorems 1, 2 and 3). Our method is based upon Selberg's original papers[5][6], and Shapiro's[7]. The umbral calculus is very effective in our description of the calculations and results[1]. The results of our previous paper[2] are used here without proofs.

2. Preliminary Lemmas and Notions

Lemma 1. For every integers k and i, the number theoretic function $[k]^i\geqq 0$ with the following initial conditions: $k\geqq 0$, $k\geqq i$, $[0]^i=1$ for $i=0,1$, $1/|i|!$ for $i<0$ and $[k]^i=0$ for $k<i$, is defined by the recurrence formula $[k]^i=[k-i]^i+i[k-1]^{i-1}$. Then, we get $[k]^i=k!/(k-i)!$ $(i\lesseqgtr 0)$. $[k]^i\,(i\gtreqless 0)$. is said the factorial polynomial in k degree i.

Lemma 2.

$$\sum_{i=l+m}^{k}(-1)^i[i]^m\binom{k}{i}\binom{i-m}{l}=\begin{cases}0, & \text{for } k\neq l+m,\\ (-1)^k[k]^{k-l}, & \text{for } k=l+m,\end{cases}$$

where $\binom{k}{i}=[k]!/i!$, $k\geqq i\geqq 0$ is the binomial coefficient.

Lemma 3. λ_n is a partition of n and if there are m_1 parts equal to 1, m_2 parts equal to 2, m_3 parts equal to 3, etc., then the partition may be written as[4] $\lambda_n=(1^{m_1}2^{m_2}3^{m_3}\ldots)$, $m_i\geqq 0$, and we put $m=\sum_{i=1}^{m}m_i$, $p(\lambda_n)=m!/m_1!\ m_2!\ldots m_n!=\binom{m}{m_1,m_2,\cdots m_n}$. We associate a monomial $M(\lambda_n,x)=M(\lambda_n,x_1,\ldots,x_n)=x_1^{m_1}x_2^{m_2}\ldots x_n^{m_n}$ with a partition λ_n. Put $A^n=A^n(x)=A^n(x_1,x_2,\ldots,x_n)=\sum_{\lambda_n}p(\lambda_n)M(\lambda_n,x)$, then, we have $A^n=\sum_{j=1}^{n}x_jA^{n-j}$.