112. On Completeness of Uniform Spaces

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(Comm. by K. KUNUGI, M.J.A., Nov. 12, 1953)

Let R be an abstract space. For a system of mappings $\mathfrak{a}_\lambda$ of R into uniform spaces $S_\lambda (\lambda \in \Lambda)$, the weakest uniformity on R for which all $\mathfrak{a}_\lambda (\lambda \in \Lambda)$ are uniformly continuous, is called the *weak uniformity of R* by $\mathfrak{a}_\lambda (\lambda \in \Lambda)$. Concerning the completeness of the weak uniformity we have[1)]

Theorem I. *Let the uniformities $\mathfrak{U}_\lambda$ of $S_\lambda (\lambda \in \Lambda)$ be separative and complete. In order that the weak uniformity of R by a system of mappings $\mathfrak{a}_\lambda$ of R into $S_\lambda (\lambda \in \Lambda)$ be complete, it is necessary and sufficient that for a system of points $x_\lambda \in S_\lambda (\lambda \in \Lambda)$ if*

$$\prod_{\nu=1}^{n} \mathfrak{a}_{\lambda_\nu}^{-1} (U_{\lambda_\nu}(x_{\lambda_\nu})) \neq 0$$

for every finite number of elements $\lambda_\nu \in \Lambda$ and $U_{\lambda\nu} \in \mathfrak{U}_{\lambda\nu} (\nu=1,2,\ldots,n)$, then we can find a point $x \in R$ for which $\mathfrak{a}_\lambda(x) = x_\lambda$ for every $\lambda \in \Lambda$.

The purpose of this paper is to give some generalization of this Theorem I and its applications.

I

For a uniform space R with uniformity $\mathfrak{V}$, a system of mappings $\mathfrak{a}_\gamma (\gamma \in \Gamma)$ of R into a uniform space S with uniformity $\mathfrak{U}$ is said to be *equi-continuous*, if for any $U \in \mathfrak{U}$ we can find $V \in \mathfrak{V}$ such that

$$\mathfrak{a}_\gamma (V(x)) \subset U(\mathfrak{a}_\gamma(x)) \qquad \text{for every } x \in R \text{ and } \gamma \in \Gamma.$$

With this definition we have

Theorem II. *Let the uniformity $\mathfrak{U}_\lambda$ of $S_\lambda (\lambda \in \Lambda)$ be separative and complete. For a double system of mappings $\mathfrak{a}_{\gamma,\lambda}$ of an abstract space R into $S_\lambda (\gamma \in \Gamma_\lambda, \lambda \in \Lambda)$, there exists the weakest uniformity on R for which $\mathfrak{a}_{\gamma,\lambda} (\gamma \in \Gamma_\lambda)$ is equi-continuous for every $\lambda \in \Lambda$, and in order that this uniformity on R be complete, it is necessary and sufficient that for a system of points $x_{\gamma,\lambda} \in S_\lambda (\gamma \in \Gamma_\lambda, \lambda \in \Lambda)$ if*

$$\prod_{\nu=1}^{n} \prod_{\gamma \in \Gamma_{\lambda_\nu}} \mathfrak{a}_{\gamma,\lambda_\nu}^{-1} (U_{\lambda_\nu}(x_{\gamma,\lambda_\nu})) \neq 0$$

for every finite number of elements $\lambda_\nu \in \Lambda$ and $U_{\lambda\nu} \in \mathfrak{U}_{\lambda\nu} (\nu=1,2,\ldots,n)$, then we can find a point $x \in R$ such that

$$x_{\gamma,\lambda} = \mathfrak{a}_{\gamma,\lambda}(x) \qquad \textit{for all } \gamma \in \Gamma_\lambda, \lambda \in \Lambda .$$

1) H. Nakano: Topology and linear topological spaces, Tokyo Math. Book Ser. II, Tokyo (1951), § 35 Theorem 8. In the present paper we make use of terminologies and notations in this book. This book will be denoted by TLTS.