108. *Structure of a Riemann Space*

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Let V_n be an n-dimensional Riemann space with positive definite line element

$$ds^2 = g_{ij}(x)\,dx^i\,dx^j \quad (i, j = 1, 2, \dots, n)$$

in each of its coordinate neighborhoods. Let Γ^i_{jk} be the Christoffel symbols of the second kind made by g_{ij}.

$$R^i_{jhk} = \frac{\partial \Gamma^i_{jk}}{\partial x^h} - \frac{\partial \Gamma^i_{jh}}{\partial x^k} - \Gamma^m_{jh}\Gamma^i_{mk} + \Gamma^m_{jk}\Gamma^i_{mh},$$

$$R_{jh} = R^k_{jhk}, \quad R = g^{ij} R_{ij}$$

are the components of the Riemann-Christoffel tensor, the Ricci tensor and the scalar curvature of the space.

In a previous paper[1], the author has investigated the spaces whose Ricci tensors satisfy the conditions

$$\text{(a)} \qquad R^k_i R^j_k = \frac{1}{n-1} R R^j_i,$$

$$\text{(b)} \qquad R^j_{i,k} = 0$$

where a comma "," denotes the covariant differentiation of the spaces. The first of these conditions is analogous to the condition for Einstein spaces, i.e.

$$R^j_i = \frac{1}{n} R\,\delta^j_i \quad (\delta^j_i = 1 \text{ or } 0, \text{ as } i = j \text{ or } i \neq j)$$

which is equivalent to the condition

$$R^k_i R^j_k = \frac{1}{n} R R^j_i.$$

In connection with Theorem 4 in the paper above, we shall prove a more precise theorem as follows:

Theorem. *If a Riemann space satisfies the conditions* (*a*), (*b*), *then it is an Einstein space with zero scalar curvature or a product space with an Einstein space* (*a surface of constant curvature*) *and a straight line.*

Proof. Let us put

$$\text{(1)} \qquad W_{ij} = R_{ij} - \frac{1}{n-1} R\,g_{ij}.$$

Then we have

$$W = g^{ij} W_{ij} = R - \frac{n}{n-1} R = -\frac{1}{n-1} R,$$

1) T. Ōtsuki: On Some Riemann Spaces, Math. J. Okayama University, Vol. 3, No. 1, pp. 65–88.