# 14. Two Remarks on Dimension Theory for Metric Spaces

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The purpose of this brief note is to make slight remarks on extensions of the well-known theorems in dimension theory for metric spaces.

First, we can extend Eilenberg-Otto's theorem to the countable dimensional case as follows.

**Proposition 1.** *A metric space $R$ is countable-dimensional, i.e. it is represented as a countable sum of 0-dimensional spaces if and only if for every collections $\{U_i \mid i=1,2,\cdots\}$ of open sets and $\{F_i \mid i=1,2,\cdots\}$ of closed sets satisfying $F_i \subset U_i$, $i=1,2,\cdots$, there exists a collection $\mathfrak{V}=\{V_i \mid i=1,2,\cdots\}$ of open sets such that*

(1) $F_i \subset V_i \subset U_i, \quad i=1,2,\cdots$

(2) *$\{B(V) \mid V \in \mathfrak{V}\}$ is point-finite, i.e. its order is finite at every point $p$ of $R$, where $B(V)$ denotes the boundary of $V$.*

*Proof.* Since the "only if" part is a direct consequence of [1, Theorem 2], we show only the "if" part. By R. H. Bing's theorem [2] we can find a $\sigma$-discrete basis $\mathfrak{U}=\bigcup_{i=1}^{\infty}\mathfrak{U}_i$ for the metric space $R$. Let $\mathfrak{U}_i=\{U_\gamma \mid \gamma \in \Gamma_i\}$, $U_\gamma=\bigcup_{j=1}^{\infty}F_{\gamma j}$ for closed sets $F_{\gamma j}$. Furthermore, let $U_i=\bigcup\{U_\gamma \mid \gamma \in \Gamma_i\}$, $F_{ij}=\bigcup\{F_{\gamma j} \mid \gamma \in \Gamma_i\}$. Then, since $F_{ij} \subset U_i$, $i,j=1,2,\cdots$, we can find a collection $\mathfrak{V}=\{V_{ij} \mid i,j=1,2,\cdots\}$ of open sets such that $F_{ij} \subset V_{ij} \subset U_i$, $\{B(V) \mid V \in \mathfrak{V}\}$ is point-finite. Letting $V_{ij} \cap U_\gamma = W_{\gamma j}$, $\gamma \in \Gamma_i$ we get a locally finite collection $\mathfrak{W}_{ij}=\{W_{\gamma j} \mid \gamma \in \Gamma_i\}$. Now $\mathfrak{W}=\bigcup\{\mathfrak{W}_{ij} \mid i,j=1,2,\cdots\}$ is a $\sigma$-locally finite basis of $R$ such that $\{B(W) \mid W \in \mathfrak{W}\}$ is point-finite. Hence by [1, Theorem 1], we can conclude that $R$ is countable-dimensional.

Next, we can give an extension to the sum-theorem as follows.

**Proposition 2.** *Let $\{F_\alpha \mid \alpha<\tau\}$ be a covering of a metric space $R$ consisting of subsets $F_\alpha$ with $\dim F_\alpha \leqq n$, $\alpha<\tau$ such that $\{F_\alpha \mid \alpha<\beta\}$ is closed for every $\beta<\tau$. Then $\dim R \leqq n$.*

*Proof.* E. Michael gave a simple proof of this theorem by use of the sum-theorem for countably many closed sets and locally finite collection of closed sets which is due to K. Morita [3] and partly to M. Katětov [4] and the others. Now, however, let us give a sketch of a direct proof. We assume $F_\alpha \cap F_\beta=\phi$ for every $\alpha, \beta$ with $\alpha \neq \beta$ without loss of generality.

In the case of $n=0$, let $G$ and $H$ be disjoint closed sets of $R$. Then we can define, by induction with respect to $\alpha$,