# 13. On Rings of Continuous Functions and the Dimension of Metric Spaces

By Jun-iti NAGATA

Osaka City University and University of Washington

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M. Katětov [1] has once established an interesting theory on a relation between the inductive (Menger-Urysohn) dimension of a compact space $R$ and the structure of the ring of all continuous functions on $R$. The purpose of this brief note is to give a slight extension to Katětov's theory for a metric space while simplifying his discussion.

According to [1], we consider an *analytical ring*, i.e. a commutative topological ring with a unit $e$ and a continuous real scalar multiplication. A subring $C_1$ of an analytical ring $C$ is called *analytically closed* if

(1) $\lambda e \in C_1$ for any real $\lambda$, (2) $x \in C_1$ whenever $x \in C$, $x^n + a_1 x^{n-1} + \cdots + a_n = 0$, $a_i \in C_1$, (3) $\bar{C}_1 = C_1$.

Let $C'$ be a subset of $C$; then a subset $M$ of $C$ is called an analytical base of $C'$ in $C$ if there exists no analytically closed subring $C_1 \not\supset C'$ containing $M$. The least number of an analytical base of $C'$ in $C$ is called the analytical dimension of $C'$ in $C$ and denoted by $\dim(C', C)$. The ring $C(R)$ of all bounded real-valued continuous functions of $R$ is an analytical ring as for its strong topology. We denote by $U(R)$ the subset of $C(R)$ consisting of all uniformly continuous functions. Furthermore, according to [2], we call a continuous mapping $f$ of a metric space $R$ into a metric space $S$ *uniformly* 0-*dimensional* if for any $\varepsilon > 0$ there exists $\eta > 0$ such that $\delta(U) < \varepsilon$ whenever $U \subset R$, diam $f(U) < \eta$, where $\delta(U) < \varepsilon$ means the fact that there exists an open covering $\mathfrak{V}$ of $U$ such that mesh $\mathfrak{V} = \sup\{\operatorname{diam} V \mid V \in \mathfrak{V}\} < \varepsilon$ and order $\mathfrak{V} \leqq 1$. The covering dimension of $R$ or the strong inductive dimension of $R$ as the same is denoted by $\dim R$. Now we can prove the following

**Theorem.** $\dim R = \dim(U(R), C(R))$ *for every locally compact, metric space* $R$.

To establish this theorem we prove some lemmas.

**Lemma 1.** *Let* $f(x) = (f_1(x), \cdots, f_n(x))$ *be a uniformly* 0-*dimensional, bounded mapping of a metric space* $R$ *into the* $n$-*dimensional Euclidean space* $E_n$. *Let* $C_1$ *be an analytically closed subring of* $C(R)$ *containing* $f_1, \cdots f_n$; *then for every sets* $F$ *and* $G$ *of* $R$ *with distance* $(F, G) = d(F, G) > 0$, *there exists* $g \in C_1$ *such that* $g(F) \geqq 1$, $g(G) = 0$, *where* $g(F) \geqq 1$, *for example, means that* $g(x) \geqq 1$ *for every* $x \in F$.