92. On the Equivalence of Excessive Functions and Superharmonic Functions in the Theory of Markov Processes. II

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1. **Introduction.** Let X be a strict Markov process on a locally compact Hausdorff space.[1] Consider a family $\mathcal{F}_x$ of admissible subsets which depends on the state $x \in S$ and put $\mathcal{F}=\bigcup_{x \in S}\mathcal{F}_x$. A nonnegative and $\mathcal{B}$-measurable function u is called *$\mathcal{F}$-superharmonic* if

(1.1) $\qquad u(x) \geqq H_{\hat{\sigma}_A} u(x) \quad$ for every $A \in \mathcal{F}_x$,

and *$\mathcal{F}$-continuous* if

(1.2) $H_{\hat{\sigma}_{A_n}} u(x) \rightarrow u(x)$ for any A_n in $\mathcal{F}_x$ such that $P_x\{\hat{\sigma}_{A_n} \downarrow 0\}=1$,

where $\hat{\sigma}_A$ is the positive hitting time for the set A, that is, $\hat{\sigma}_A = \inf\{t>0, x_t \in A\}$.[2] In these terminologies, a nonnegative superharmonic function u in the usual sense is caught in the following way. Let S be an open domain of R^n and X, the Brownian motion on S. Consider the euclidean metric ρ and denote the ball $\{y; \rho(x, y)<r\}$ by $U_{x,r}$. A nonnegative function u is superharmonic in the classical sense if and only if it is $\mathcal{F}$-superharmonic and $\mathcal{F}$-continuous with respect to the family $\mathcal{F}$ induced by $\mathcal{F}_x=\{\overline{U}^c_{x,r};\ r>0$ and $\overline{U}_{x,r}$ is compact$\}$.[3]

According to Proposition 2.4 of (I), any excessive function is $\mathcal{F}$-superharmonic and $\mathcal{F}$-continuous for any family $\mathcal{F}$. The converse problem is now stated as follows: *For what family $\mathcal{F}$ does it hold that every $\mathcal{F}$-superharmonic and $\mathcal{F}$-continuous function is excessive?* This problem is solved for a sufficiently large $\mathcal{F}$, affirmatively. For example, a theorem due to Dynkin [1][4] asserts that it is enough to take the family $\mathcal{F}$ such that $\mathcal{F}_x=\{$every compact sets in $S\}$, if X satisfies the quasi-continuity from the left.[5] But this theorem seems

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1) In this paper we shall use the terminologies and notations of the previous paper [4] with no special reference. In the following, it will be quoted as (I).

2) Since A is admissible, the nonnegative hitting time $\sigma_A=\inf\{t \geqq 0, x_t \in A\}$ is a Markov time. Noting that $t+\sigma_A(w_t^+) \downarrow \hat{\sigma}_A(w)$, $\hat{\sigma}_A$ is also a Markov time.

3) In this case, it is shown that the $\mathcal{F}$-continuity with respect to our family is equivalent to the lower semicontinuity, using special properties of the Brownian motion.

4) He stated this theorem without proof. Recently, Motoo has proved it (private communication).

5) For the definition, see [2]. It is known that any Borel set (and therefore any compact set) of S is admissible if X is quasi-continuous from the left.