138. *On Concircular Scalar Fields*

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In a previous paper [1], the author determined complete Riemannian manifolds admitting a concircular scalar field [1, Theorem 1] and, in particular, those admitting a special concircular scalar field [1, Theorem 2]. If M is an n-dimensional Riemannian manifold with metric tensor field $g_{\mu\lambda}$, and we denote by $\{{}^{\kappa}_{\mu\lambda}\}$ the Christoffel symbol and by ∇ the covariant differentiation with respect to $\{{}^{\kappa}_{\mu\lambda}\}$, then a *concircular scalar field* ρ is by definition a scalar field satisfying the differential equation

(1) $$\nabla_\mu \nabla_\lambda \rho = \phi g_{\mu\lambda} ,$$

where ϕ is a scalar field and will be called the *characteristic function* of ρ. A *special* concircular scalar field is by definition a concircular one satisfying

(2) $$\nabla_\mu \nabla_\lambda \rho = (-k\rho + b) g_{\mu\lambda}$$

with constant coefficients k and b. We shall call k the *characteristic constant* of ρ.

In the present paper we shall show that, if a Riemannian manifold admits functionally independent concircular scalar fields, then they are special concircular scalar fields having the same characteristic constant, and that a concircular scalar field, which is not invariant under an infinitesimal isometry, is also a special concircular one. In this light, we may say that the special concircular scalar field is *not* so special.

A point is called a stationary or ordinary point of ρ according as the gradient vector field $\rho_\lambda = \partial_\lambda \rho$ vanishes there or not. We notice that the characteristic function ϕ of ρ is a differentiable function of ρ itself in a neighborhood of an ordinary point of ρ. We shall first show the following

Lemma 1. *If ρ and σ are concircular scalar fields functionally dependent of each other, then σ is linear in ρ with constant coefficients in a neighborhood of an ordinary point of ρ.*

Proof. Suppose that σ satisfies the equation

(3) $$\nabla_\mu \sigma_\lambda = \psi g_{\mu\lambda} ,$$

where $\sigma_\lambda = \partial_\lambda \sigma$. By our assumption, we may put $\sigma_\lambda = A\rho_\lambda$, A being a proportional factor. By substituting this into (3) and using (1), we have

$$(\partial_\mu A)\rho_\lambda = (\psi - A\phi) g_{\mu\lambda} .$$