224. *On Imbeddings and Colorings of Graphs. II*

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§1. Introduction. In this paper we use the same definitions and notations as in the part I. The following theorem is proved in this paper.

Theorem (1.1). *If there exists an m-imbeddable and n-chromatic graph, there exists a graph of which genus is m′ and the chromatic number is n′, for any m′ and n′ satisfying $m'\geqq m$ and $2\leqq n'\leqq n$.*

In §2 we prove theorem (2.5), which is more general than theorem (1.1), by using part I. But theorem (1.1) can be proved directly by the idea which professor Y. Saito told me. An outline of this idea is shown at the end of this paper.

Lemma (2.1). *Let the chromatic number of H be larger than 2. If there is an imbedding of H into M, there is a graph G such that (i) H and G have the same chromatic number, (ii) G has no k-circuit for $k<3$, and (iii) there is a 2-cell imbedding $G(M)$.*

Proof. Let $H(M)$ be the given imbedding, and let α be any component of M-$H(M)$. If H is n-chromatic, there is a color-classification $H^0=\gamma_1\cup\cdots\cup\gamma_n$.

We can take arcs $a_1,\cdots,a_p$ in α joining the vertices in $\mathrm{Bd}\,\alpha$ and not intersecting each other such that any component of $\alpha-a_1\cup\cdots\cup a_p$ is an open 2-cell. Let A_i, B_i be the edges of a_i. Here we permit happening $A_i=B_i$. Let C_i be the center of a_i. Then, we construct as follows an n-chromatic graph H_α imbedding $H_\alpha(M)$ and color-classes $\gamma_{\alpha,1},\cdots,\gamma_{\alpha,n}$:

(i) $H_\alpha^0=H^0\cup\{C_1,\cdots,C_p\}$,
$H_\alpha^1=H^1\cup\{(A_i,c_i),(B_i,C_i)\mid i=1,\cdots,p\}$.

(ii) $H_\alpha(M)\mid H=H(M)$, $H(M)(C_i)=C_i$ and $H_\alpha(M)\mid(A_i,C_i)\cup(B_i,C_i)$ is onto a_i.

(iii) As the chromatic number of $H\geqq 3$, for a_i we can fix γ_j such that $\gamma_j\neq\gamma(A_i),\gamma(B_i)$. If we note this γ_j by $\gamma(a_i)$, we have the color-classes of H_α as follows:

$$\gamma_{\alpha,j}=\gamma_j\cup\{C_i\mid\gamma(a_i)=\gamma_j\},\ (j=1,\cdots,n).$$

We repeat the same modification for all non cellular components of $M-H(M)$, and finally we obtain H_1, $H_1(M)$ and color-classification $H^0=\gamma_{1,1}\cup\cdots\cup\gamma_{1,n}$. H_1 satisfies the conditions (i) and (iii) which G is to satisfy.

Next, let $a_1,\cdots,a_q$ be all the 1-circuits contained in H_1 and let A_i be the vertex on a_i, namely $a_i=(A_i,A_i)\in H_1^1$. To take away