258. On Fractional Powers of the Stokes Operator

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1. Introduction and summary. The present paper is concerned with the so-called Stokes operator described below. Our objective is to prove a theorem concerning domains of fractional powers of the Stokes operator. This theorem has some applications to the Navier-Stokes equation [4], as is expected from important roles played by the fractional powers of the Stokes operator in recent works on the Navier-Stokes equation. For instance, see Sobolevskii [11, 12], Kato-Fujita [7], Fujita-Kato [3], and Masuda [10]. Moreover, we hope that the theorem is of some interests also from the view point of theory of fractional powers of operators and theory of interpolation of spaces.

Let Ω be a bounded domain in R^m with smooth boundary $\partial\Omega$. By L we denote $L_2(\Omega)$ of real m-vector functions defined in Ω. $C_{0,\sigma}^\infty$ is the set of all vector functions $\varphi \in C^\infty(\Omega)$ with $\operatorname{div}\varphi=0$ and $\operatorname{supp}\varphi\subset\Omega$. We put

$$H_\sigma=\text{the closure of } C_{0,\sigma}^\infty \text{ in } L_2(\Omega),$$
$$H_\sigma^1=\text{the closure of } C_{0,\sigma}^\infty \text{ in } W_2^1(\Omega).$$

Here, $W_2^l(\Omega)$ means the Sobolev space of order l. The orthogonal projection from L onto H_σ is denoted by P. The operator $A_0=-P\Delta$ with domain $C_{0,\sigma}^\infty$ is positive and symmetric in the Hilbert space H_σ. The Friedrichs extension A of A_0 is called the *Stokes operator* in Ω. A is positive and self-adjoint. It should be noted that $Au=Pf$ $(f\in L)$ implies that

$$(1.1)\qquad \begin{cases} \Delta u-\nabla p=-f & \text{in } \Omega,\\ \operatorname{div} u=0 & \text{in } \Omega,\\ u|_{\partial\Omega}=0 \end{cases}$$

with some scalar function p. Actually, it is known [2, 8] that

$$(1.2)\qquad \mathcal{D}(A)=W_2^2(\Omega)\cap H_\sigma^1,$$

where $\mathcal{D}(A)$ is the domain of the operator A. On the other hand, we put $B=-\Delta$ with

$$(1.3)\qquad \mathcal{D}(B)=W_2^2(\Omega)\cap H^1,$$

where H^1 is the set of all $u\in W_2^1(\Omega)$ satisfying $u|_{\partial\Omega}=0$. Obviously, B is a positive self-adjoint operator in L.

Our theorem now reads:

Theorem 1.1. *Let A and B be as above. Then for any α in $0<\alpha<1$, we have*