

245. A Note on Compactness and Metacompactness

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(Comm. by Kinjirô KUNUGI, M. J. A., Dec. 12, 1970)

The purpose of this paper is to give a short proof to a theorem by G. Aquaro [2] which states that a space X is compact if and only if it is countably compact and each open cover of X has a point countable open refinement. Arens and Dujundji used Zorn's lemma to prove that a T_1 space is compact if and only if it is metacompact and countably compact [1].

An open cover of a space X is said to be a point (finite) countable open cover of X if each point in the space is in only a (finite) countable number of members of the open cover [3].

Theorem 1. *If $C = \{G_\alpha \mid \alpha \in A\}$ is a point (finite) countable open cover of a space X which has no (finite) countable subcover, then there exists a sequence $\{x_n\}$ such that if $i \neq j$ then $\{x_i, x_j\} \cap G_\alpha = \phi$ for each $\alpha \in A$.*

Proof. Let $G_{(\alpha,1)} \in C$ and let $x_1 \in G_{(\alpha,1)}$. There exists $x_2 \in X - \cup\{G_\alpha \mid G_\alpha \in C \text{ and } x_1 \in G_\alpha\}$. For each positive integer n there exists $x_n \in X - \cup\{G_\alpha \mid x_i \in G_\alpha \text{ for some } i < n\}$. The constructed sequence $\{x_n\}$ is a desired sequence for the theorem.

Remark. It is immediate that the sequence in Theorem 1 has no cluster point. If it did then the cluster point would be in some $G_\alpha \in C$ which would result in a contradiction.

By the fact that a space X is countably compact if and only if each sequence has a cluster we now have the following corollaries. The first corollary was the key result in a paper by G. Aquaro [2].

Corollary 1. *If X is a countably compact topological space then for each point countable open covering of X there exists a finite subcovering.*

Corollary 2. *A space X is compact if and only if it is countably compact and each open cover of X has a point countable open refinement.*

The following corollary with the added condition of T_1 was first proven by Arens and Dujundji [1].

Corollary 3. *A space X is compact if and only if it is metacompact and countably compact.*