220. A Remark on the Spectral Order of Operators

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1. **Spectral order.** Consider a Hilbert space $\mathfrak{H}$. Let $\mathcal{S}$ be the set of all hermitean (bounded linear) operators acting on $\mathfrak{H}$. Suppose that $A, B \in \mathcal{S}$. Then the spectral theorem assures us that there are the spectral resolutions of the identity $E(t)$ and $F(t)$ such as

$$(1) \qquad A=\int_{-\infty}^{\infty} tdE(t) \quad \text{and} \quad B=\int_{-\infty}^{\infty} tdF(t).$$

Very recently, besides the usual order of hermitean operators, M. P. Olson [4] introduce a new order in the following manner:

Definition A. If A and B are hermitean operators as in (1), then $A \preceq B$ provided that

$$(2) \qquad E(t) \geqq F(t) \qquad \text{for } -\infty < t < \infty.$$

This new order is called the *spectral order* by Olson. He proves, among others, that the spectral order differs from the usual one (coincides if A commutes with B) and $\mathcal{S}$ becomes a conditionally complete lattice under the spectral order.

In the present note, we shall introduce an another definition of the spectral order (cf. [4, Cor. 1]):

Definition B. If A and B are hermitean operators, then $A \ll B$ if and only if

$$(3) \qquad f(A) \leqq f(B)$$

for any continuous monotone (nondecreasing) function f defined on an interval which contains $\sigma(A) \cup \sigma(B)$, where $\sigma(C)$ is the spectrum of C.

In the below, we shall reconstruct Olson's theory under Definition B.

2. **Properties.** The following theorem is our main result:

Theorem 1. *Definitions A and B are equivalent.*

Proof. Suppose that $A \preceq B$ in the sense of Olson. Suppose furthermore that A and B are expressed in (1). Then $E(t) \geqq F(t)$ for every t. If f is a continuous monotone function, then we have

$$\begin{aligned}(f(A)x \mid x) &= \int_a^b f(t)d(E(t)x \mid x) \\ &= f(b)\|x\|^2 - \int_a^b \|E(t)x\|^2 \, df(t) \\ &\leqq f(b)\|x\|^2 - \int_a^b \|F(t)x\|^2 \, df(t)\end{aligned}$$