171. *On the Regularity of Domains for the First Boundary Value Problem for Semi-linear Parabolic Partial Differential Equations*

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In this short note, we shall prove that a domain $D \subset R^{n+1}$ is regular for the first boundary value problem (=the Dirichlet problem or the initial-boundary value problem) for the semi-linear parabolic partial differential equation:

$$\text{(E)} \qquad \boldsymbol{P}u \equiv \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} - \frac{\partial u}{\partial t} = f\left(x, t, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right),$$

if it is regular for $\boldsymbol{P}u \leqq -1$.

It is well known that even for the simplest equation of this kind, namely, for the heat equation

$$\text{(H)} \qquad \triangledown u \equiv \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} - \frac{\partial u}{\partial t} = 0,$$

there may not be a solution u for the first boundary value problem if we require u to take values prescribed on the (whole) topological boundary of the domain. For example, consider the first boundary value problem for (H) for $n=1$ for the domain $\{(x, t); 0<x<1, 0<t<1\}$. Values of the solution $u(x,t)$ on the upper boundary $\{(x,t); 0 \leqq x \leqq 1, t=1\}$ are determined by the values of u given on the side boundary $\{(x,t); x=0 \text{ or } 1, 0 \leqq t \leqq 1\}$ and the lower boundary $\{(x,t); 0 \leqq x \leqq 1, t=0\}$.

Prompted by this example, let us split the topological boundary ∂D of a domain D bounded by a finite number of sufficiently smooth hypersurfaces into three parts, namely, i) Side boundary $\partial_s D$: closure of the part where the outer normal is not parallel to the time axis, ii) Lower boundary $\partial_l D$: closure of the part where the outer normal is in the $-t$ direction, and iii) Upper boundary $\partial_u D$: interior of the part where the outer normal is in $+t$ direction. We shall call the set $\partial_p D \equiv \partial_l D \cup \partial_s D$ the *parabolic boundary* of D, which is the set where we should give the boundary data. In other words, a point of $\partial_u D$ must be considered parabolically an interior point of D. So, the question to be asked will be: is there always a solution of $\triangledown u = 0$ (or more generally, $\boldsymbol{P}u = f$) in D admitting a continuous boundary value prescribed on $\partial_p D$?

Another example shows that there is not always a solution. Let $C(P, r)$ be the parabolic circle (sphere) for the heat equation (H) for