4. On the Integration of the Temporally Inhomogeneous Diffusion Equation in a Riemannian Space

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1. Introduction. Let R be a connected domain of an m-dimensional, orientable C^∞ Riemann space with the metric $ds^2 = g_{ij}(x)dx^i dx^j$. We consider the forward diffusion equation in R

(1. 1) $$E_{tx}f = \frac{\partial f(t, x)}{\partial t} - A_{tx}f(t, x) = 0,\ t>s\,,$$

where

(1. 2) $$A_{tx}f(t, x) = g(x)^{-1/2} \frac{\partial^2}{\partial x^i \partial x^j}(g(x)^{1/2}a^{ij}(t, x) f(t, x)) - g(x)^{-1/2}\frac{\partial}{\partial x^i}(g(x)^{1/2} b^i(t, x) f(t, x)) + c(t, x) f(t, x)\,,$$

$$g(x) = \det(g_{ij}(x))\,.$$

The associated backward diffusion equation is defined by

(1. 3) $$E^*_{sy}h = -\frac{\partial h(s, y)}{\partial s} - A^*_{sy}h(s, y) = 0,\ s<t\,,$$

where A^*_{sy} is the formal adjoint of A_{ty}:

(1. 4) $$A^*_{sy}h(s,y) = a^{ij}(s,y)\frac{\partial^2 h(s,y)}{\partial y^i \partial y^j} + b^i(s,y)\frac{\partial h(s,y)}{\partial y^i} + c(s,y)h(s,y).$$

The operator $A_t = A_{tx}$ is assumed to be elliptic in x in the sense that

(1. 5) $$a^{ij}(t,x)\xi_i\xi_j > 0 \text{ for } \sum_i (\xi_i)^2 > 0\,.$$

Since the value of $A_{tx}f(t,x)$ should be independent of the local coordinates $(x^1, \ldots, x^m)$, we must have, by the coordinates change $x \to \bar{x}$, the transformation rule

(1. 6) $$a^{ij}(t,\bar{x}) = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial \bar{x}^j}{\partial x^n} a^{kn}(t,x)\,,$$

$$\bar{b}^i(t, \bar{x}) = \frac{\partial \bar{x}^i}{\partial x^k} b^k(t, x) + \frac{\partial^2 \bar{x}^i}{\partial x^k \partial x^n} a^{kn}(t, x)\,.$$

For the sake of simplicity, we assume that the coefficients $a^{ij}(t, x)$, $b^i(t, x)$, $c(t, x)$ and $g_{ij}(x)$ are C^∞ functions of (t, x).

The purpose of the present note is to give a sketch of a method[1)]

1) Another method was proposed by Tosio Kato: (Integration of the equation of evolution in a Banach space, J. Math. Soc. Japan, **5**, 208–234 (1953)). His method is much general and elegant. However, it may not be easy to apply his method to the concrete equation such as (1.1), since he assumes that the domain $D(\bar{A}_t)$ of the closed extention $\bar{A}_t$ of A_t is independent of t.