19. On the Family of the Solution-Curves of the Integral Inequality

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A certain generalization of the theorem of Kneser on the differential inequality was shown by Prof. M. Hukuhara.[1] In this note, we shall generalize it to the case of integral inequality

$$(1) \qquad |u(x)-f(x)-\int_0^x K(x, t, u(t))dt| \leq p(x)$$

where the functions f, u and K represent n-dimensional vectors, while x, t and p are real; $f(x)$ is continuous in $0\leq x\leq 1$, $K(x, t, u)$ is bounded and continuous in the domain D:

$$0\leq t\leq x\leq 1, \qquad |u|<\infty,$$

$p(x)$ is continuous in the interval $0\leq x\leq 1$.

Suppose that the family $\mathfrak{F}$ *of* $f(x)$ *is a compact continuum in* (C) *and* $\mathfrak{U}$ *is the family of the totality of the solution-curves*[2] *of* (1) *with* $f(x)\in\mathfrak{F}$. *Then,* $\mathfrak{U}$ *is also a compact continuum in* (C).

cf. (C) denotes the space of continuous functions on $0\leq x\leq 1$ with the norm $\|f\|=\max\limits_{0\leq x\leq 1}|f(x)|$.

It is evident that the family $\mathfrak{U}$ is a closed and compact set in (C). If $\mathfrak{U}$ is not a continuum, $\mathfrak{U}$ must be the sum of two closed, disjoint and non void sets $\mathfrak{U}_1$ and $\mathfrak{U}_2$. Let $\mathfrak{F}_i$ be the family of the functions $f_i(x)$ whose corresponding solutions are in $\mathfrak{U}_i (i=1, 2)$. Then $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are closed and $\mathfrak{F}=\mathfrak{F}_1\cup\mathfrak{F}_2$. As $\mathfrak{F}$ is a continuum, there exists f_0 such that

$$f_0\in\mathfrak{F}_1\cap\mathfrak{F}_2.$$

The family $\mathfrak{U}_0$ of the solution-curves corresponding to f_0 contains an element of $\mathfrak{U}_1$ and an element of $\mathfrak{U}_2$. Therefore, if we can prove that $\mathfrak{U}_0$ is a continuum, $\mathfrak{U}_0$ must contain an element which does not belong to $\mathfrak{U}$. This contradicts to $\mathfrak{U}_0\subseteq\mathfrak{U}$. Therefore, it is sufficient to prove that $\mathfrak{U}_0$ is a continuum, i.e. the solution-curves $\mathfrak{U}_0$ of the following integral inequality

$$(2) \qquad |u(x)-f(x)-\int_0^x K(x, t, u(t))dt| \leq p(x)$$

1) M. Hukuhara: Sur une généralisation d'un théorème de Kneser, Proc. Japan Acad., **29**, 154 (1953).

2) 3) For the existence of such solutions, see T. Satô's "Sur les équations integrales non-linéaires de Volterra" (forthcoming in «Compositio Mathematica»).