17. On Ideals in Rings of Continuous Functions

By Taira SHIROTA

Osaka University, Osaka

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In the theory of function rings the boundedness of functions or the compactness of the base spaces plays a very important rôle, but it seems to be necessary to remove the condition of the boundedness or the compactness.

In the present note we concern ourselves with ideals of rings of continuous functions which are not always bounded.

1. **Definition 1.** Let X be a set. Then we say that a ring $\mathfrak{R}$ consisting of complex-valued functions on X with the ordinary addition and multiplication is a *normal function ring*, if it satisfies the following conditions:

(1) (*Self-adjointness*) It contains the identity and it contains a function with its conjugate function.

(2) (*Inverse closedness*) The subset $\mathfrak{R}_p$ of all strictly positive functions with inverses in $\mathfrak{R}$ possesses the following properties:

(a) If f and g belong to $\mathfrak{R}_p$, there exists an $h \in \mathfrak{R}_p$ such that $h \leqq f$ and $h \leqq g$.

(b) If f belongs to $\mathfrak{R}_p$, there exists a $g \in \mathfrak{R}_p$ such that $g^2 \leqq f$.

(c) If $f \geqq g$, $f \in \mathfrak{R}$ and $g \in \mathfrak{R}_p$, then f has an inverse.

(3) (*Regularity*) If $\text{l.u.b.}_{x \in A} |h(x)| \lneqq \text{g.l.b.}_{x \in B} |h(x)|$ for two subsets A and B of X and for some $h \in \mathfrak{R}$, then A and B are separated by a positive function h' of $\mathfrak{R}$, i.e., $h'(A) \equiv 0$ and $h'(B) \equiv 1$.

Then we see that the concept of a normal function ring is an extension of that of the inverse closed, regular, $*$-commutative, algebra with the identity. We have many examples of normal function rings. For instance, the ring of all complex-valued continuous functions on a completely regular space X: $\mathfrak{C}(X, K)$, the ring of all complex-valued uniformly continuous functions on a uniform space X: $\mathfrak{C}_u(X, K)$, and the ring of all complex-valued r-differentiable functions on an r-differentiable manifold M^r: $\mathfrak{C}_r(M^r)$ $(0 \leqq r \leqq \infty)$ are normal function rings.

2. We introduce a natural topology in a normal function ring.

Definition 2. Let $\mathfrak{R}$ be a normal function ring with base X and let $U_\pi = \{f \mid f \in \mathfrak{R}\ \&\ |f| < \pi\}$ for some $\pi \in \mathfrak{R}_p$ where $|f|$ is the function whose value at any point x of X is the absolute value of $f(x)$. Then the m-topology of $\mathfrak{R}$ is the one with a fundamental system of neighbourhoods of 0 $\{U_\pi \mid \pi \in \mathfrak{R}_p\}$.