16. On Quasi-Translations in E^n

By Hidetaka TERASAKA
Department of Mathematics, Osaka University
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By a *quasi-translation* will be meant a sense preserving topological transformation f of a Euclidean space E^n onto itself such that for every bounded set M its iterated images $f^n(M)$ for $n\to\pm\infty$ have no cluster set, i.e.

$$\overline{\lim}_{n\to\pm\infty} f^n(M)=0,$$

or roughly speaking, $f^n(M)$ diverges to infinity when $n\to\pm\infty$.

A quasi-translation is a fortiori fixed point free and moreover regular (or singularity free) in the sense of Kerékjártó-Sperner. Thus a quasi-translation is by the theorem of Kerékjártó-Sperner[1][2] topologically equivalent to a translation in the ordinary sense if E^n is a plane. Whether or not this is true for $n\geqq 3$ remains still open. The purpose of this note is to give a simple proof of Theorem I, which may serve as a lemma to settle this question. The theorem of Kerékjártó-Sperner is an immediate consequence of our theorem.

Theorem I. *Let f be a quasi-translation of E^n. Then there is an unbounded polyhedron π such that if D denotes the domain bounded by π and $f(\pi)$, then $f^n(D)$ is disjoint from $f^m(D)$ whenever $n\neq m$, n and m being arbitrary integers, and $\bigcup_{n=-\infty}^{\infty} f^n(\overline{D})=E^n$.*

We prove the theorem in the following version, in which the sense preservation is not even assumed.

Theorem II. *Let f be a topological transformation of a sphere S^n onto itself with a single fixed point o such that if M is a set with $\overline{M}\not\ni o$, then*

$$\lim_{n\to\pm\infty} f^n(M)=o.$$

Then there exists an open polyhedron π with the sole boundary at o such that if D denotes the domain bounded by $\pi\cup o$) and $f(\pi\cup o)$, then $f^n(D)$ is disjoint from $f^m(D)$ whenever $n\neq m$, n and m being arbitrary integers, and $\bigcup_{n=-\infty}^{\infty} f^n(\overline{D})=S^n$.*

Proof. To begin with, we shall define for any set M of S^n the measure $\mu(M)$ introduced by H. Whitney[3] as follows: Let a_1, a_2, ..., a_n,... be a sequence of points dense in S^n, and put for any

*) o denotes the point o as well as the set consisting of the point o. $\pi\cup o$ means the set sum of π and o.