51. *Probabilities on Inheritance in Consanguineous Families. VII*

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VII. Mother-descendants combinations through several consanguineous marriages

1. Special combinations with several consanguineous marriages

The main purpose of the present chapter is to determine the probability of a mother-descendants combination designated by

$$\pi_{(\mu\nu;n)_t|\mu\nu}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)\equiv\overline{A}_{\alpha\beta}\kappa_{(\mu\nu;n)_t|\mu\nu}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)\qquad(\mu=\mu_{t+1},\ \nu=\nu_{t+1}).$$

By definition, the reduced probability $\kappa_{(\mu\nu;n)_t|\mu\nu}$ is given by

$$\kappa_{(\mu\nu;n)_t|\mu\nu}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)=\sum\kappa_{(\mu\nu;n)_t}(\alpha\beta;ab)\kappa_{\mu\nu}(ab;\xi_1\eta_1,\xi_2\eta_2).$$

Evidently, this probability is symmetric with respect to μ_r and ν_r for any r with $1\leqq r\leqq t$, while it is quasi-symmetric with respect to μ and ν, i.e.

$$\kappa_{(\mu\nu;n)_t|\mu\nu}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)=\kappa_{(\mu\nu;n)_t|\nu\mu}(\alpha\beta;\xi_2\eta_2,\xi_1\eta_1).$$

In the present section we first deal with the case where the n_r, μ and ν are all equal to unity. After substituting the known expressions, its defining equation yields

$$\begin{aligned}\kappa_{(\mu\nu;1)_t|11}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)&=\sigma(\xi_1\eta_1,\xi_2\eta_2)+2^{-t+1}A_tU(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)\\&+4u_t\sum R(ab)\kappa(ab;\xi_1\eta_1,\xi_2\eta_2)+2v_t\sum S(\alpha\beta;ab)\kappa(ab;\xi_1\eta_1,\xi_2\eta_2)\\&+4w_t\sum T(\alpha\beta;ab)\kappa(ab;\xi_1\eta_1,\xi_2\eta_2).\end{aligned}$$

Thus, it remains only to determine the last three residual terms, i.e.

$$\begin{aligned}\mathfrak{X}(\xi_1\eta_1,\xi_2\eta_2)&=\sum R(ab)\kappa(ab;\xi_1\eta_1,\xi_2\eta_2),\\\mathfrak{Y}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)&=\sum S(\alpha\beta;ab)\kappa(ab;\xi_1\eta_1,\xi_2\eta_2),\\\mathfrak{Z}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)&=\sum T(\alpha\beta;ab)\kappa(ab;\xi_1\eta_1,\xi_2\eta_2),\end{aligned}$$

which are evidently symmetric with respect to $\xi_1\eta_1$ and $\xi_2\eta_2$. Actual computation leads to the following results:

$$\mathfrak{X}(ii,ii)=\tfrac{1}{8}i^2(1-i)(1+i),\qquad\mathfrak{X}(ii,ik)=-\tfrac{1}{4}i^3k,$$
$$\mathfrak{X}(ii,kk)=-\tfrac{1}{8}i^2k^2,\qquad\mathfrak{X}(ii,hk)=-\tfrac{1}{4}i^2hk,$$
$$\mathfrak{X}(ij,ij)=\tfrac{1}{4}ij(1-2ij),\qquad\mathfrak{X}(ij,ik)=-\tfrac{1}{2}i^2jk,$$
$$\mathfrak{X}(ij,hk)=-\tfrac{1}{2}ijhk;$$
$$\mathfrak{Y}(ii;ii,ii)=-\tfrac{1}{16}i(1-i)^2(1-2i),\qquad\mathfrak{Y}(ii;ii,ig)=\tfrac{1}{8}ig(1-i)(1-2i),$$
$$\mathfrak{Y}(ii;ii,gg)=-\tfrac{1}{16}ig^2(1-2i),\qquad\mathfrak{Y}(ii;ii,fg)=-\tfrac{1}{8}ifg(1-2i),$$
$$\mathfrak{Y}(ii;ik,ik)=-\tfrac{1}{16}k(1-4i-k+3i^2-8i^2k),$$
$$\mathfrak{Y}(ii;ik,kk)=\tfrac{1}{16}k^2(1+i-5k+4ik),$$
$$\mathfrak{Y}(ii;ik,ig)=\tfrac{1}{16}kg(1-7i+8i^2),\qquad\mathfrak{Y}(ii;ik,kg)=\tfrac{1}{16}kg(1-3i-2k+8ik),$$
$$\mathfrak{Y}(ii;ik,gg)=-\tfrac{1}{16}kg^2(1-4i),\qquad\mathfrak{Y}(ii;ik,fg)=-\tfrac{1}{8}kfg(1-4i),$$