60. On Closed Mappings

By Sitiro HANAI
Osaka University of the Liberal Arts and Education
(Comm. by K. KUNUGI, M.J.A., April 12, 1954)

If S and E are T_1-spaces, a single-valued mapping $f(S)=E$ is said to be closed provided that the image of every closed set in S is closed in E. It is interesting to know how the topology of E is affected by the topology of S under f. Concerning this question, G. T. Whyburn and A. V. Martin have recently investigated and obtained some results.[1]

In this note, we will consider the case when the topology of E affected by the topology (under some restrictions) of S under f becomes metrizable.

1. We will firstly prove the following

Theorem 1. Let S be a perfectly separable Hausdorff space and let E a compact space.[2] If $f(S)=E$ is a closed mapping such that $f^{-1}(p)$ is compact for every point p of E, then E is a separable metric space.

To establish this theorem, we prove the following lemmas.

Lemma 1. Let S be a perfectly separable Hausdorff space. If $f(S)=E$ is a closed continuous mapping such that $f^{-1}(p)$ is compact for every point p of E, then E is perfectly separable.

Proof. Let $\{U_n\}(n=1, 2, 3, \dots)$ be a countable basis of open sets of S. For each finite subset $(n_1, n_2, \dots, n_m)$ of $(1, 2, 3, \dots)$, let $(\sum_{i=1}^{m} U_{n_i})_0$ be the union of all $f^{-1}(p)$ such that $\sum_{i=1}^{m} U_{n_i} \supset f^{-1}(p)$. Then $(\sum_{i=1}^{m} U_{n_i})_0$ is an open inverse set, and the family $\{(\sum_{i=1}^{m} U_{n_i})_0\}$ of all such sets is evidently countable.

Now let O be an open set of E and $p \in O$, then $f^{-1}(O) \supset f^{-1}(p)$ and $f^{-1}(O)$ is open in S because f is continuous. Then $f^{-1}(O)=\sum_{j=1}^{\infty} U_{n_j}$ where $\{U_{n_j}\} \subset \{U_n\}(n=1, 2, 3, \dots)$. Since $f^{-1}(p)$ is compact, there exists a finite subset $\{U_{n_k}\}(k=1, 2, \dots, l)$ of $\{U_{n_j}\}(j=1, 2, 3, \dots)$ such that $\sum_{k=1}^{l} U_{n_k} \supset f^{-1}(p)$, hence $(\sum_{k=1}^{l} U_{n_k})_0 \supset f^{-1}(p)$. As f is closed and

1) G. T. Whyburn: Open and closed mappings, Duke Math. Jour., **17**, 69–74 (1950). A. V. Martin: Decompositions and quasi-compact mappings, (abstract), Bull. Amer. Math. Soc., **59**, 397 (1953).

2) We use "compact" in the sense of "bicompact".