58. A Proof of the Plancherel Theorem

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1. **Introduction.** The purpose of this note is to give a simple proof to the generalized Plancherel theorem. This is one of the fundamental theorems in harmonic analysis on commutative topological groups, and various formulations and proofs to it have been published. We follow essentially the formulation given by Godement in his paper [2] (see the references at the end of the note); that is, we formulate the theorem as a proposition on realization of a certain kind of representation of a self-adjoint algebra. We shall give the exact statement in section 3, and the proof in section 4. The key idea of our proof is consideration of the so-called approximate identity. In section 2 we shall give definitions of some general notions which are necessary to state the theorem. In the last section, which concerns the special case of commutative groups, some remarks will be given on the Khintchine theorem on approximation of a positive definite function by the convolution of a square integrable function with its adjoint.

2. **(H)-representation.** Let $\mathbf{A}$ be a *$*$-algebra*, that is, a complex algebra which admits an adjoint operation $x \to x^*$. By a *representation* of $\mathbf{A}$ we mean a representation of the $*$-algebra $\mathbf{A}$ by bounded linear operators on a Hilbert space $\mathbf{H}$: $x \to R(x)$, where the adjoint of an operator is to be defined as the usual one. If, moreover, there corresponds to each element x in $\mathbf{A}$ an element in $\mathbf{H}$, which we designate by $\Phi(x)$ or $\dot{x}$, in such a manner that the following three conditions are satisfied, then we call the triplet $\{\mathbf{H}, R, \Phi\}$ an *(H)-representation* of $\mathbf{A}$:

(1) Φ is linear.

(2) $(xy)^{\cdot} = R(x)\dot{y}$ for $x \in \mathbf{A}$, $y \in \mathbf{A}$.

(3) The image of the whole $\mathbf{A}$ by Φ is everywhere dense in $\mathbf{H}$.

An (H)-representation is called *proper* if for any nonzero element φ in $\mathbf{H}$ there exists an x in $\mathbf{A}$ such that $R(x)\varphi \neq 0$. A filtre $\{u\}$ whose elements u belong to $\mathbf{A}$ is called an *approximate identity* with respect to the given (H)-representation, if, for each $x \in \mathbf{A}$, $\{u\dot{x}\}$ converges to $\dot{x}$ strongly in $\mathbf{H}$ according to the filtre $\{u\}$. (Every (H)-representation of $\mathbf{A}$ corresponds to a bilinear functional on $\mathbf{A} \times \mathbf{A}$ which satisfies certain simple conditions, and this correspondence may be considered as a general formulation of the method of utilizing positive functionals