56. Note on Kodaira-Spencer's Proof of Lefschetz Theorems

By Yasuo AKIZUKI and Shigeo NAKANO[1)]
Mathematical Institute, Kyoto University
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It has been for a long time expected to have a rigorous proof of the celebrated theorem of Lefschetz on the homomorphism between homology groups of an algebraic variety $V^n(n\geqq 2)$ and those of its generic hyperplane section S. Recently Kodaira and Spencer have succeeded through their deep investigations in proving not only this theorem but also the lemma of Enriques-Severi-Zariski at the same time. By a differential-geometric method Kodaira gained the fundamental lemma on the cohomology groups with coefficients in the stack of germs of holomorphic forms whose coefficients lie in a complex line bundle over V. Standing on this lemma he succeeded also in obtaining the decisive result: the Kähler variety of Hodge's type is equivalent to a projective variety (namely an algebraic variety).

The differential-geometric considerations were necessary indeed for the discovery of the lemma, but not necessary, as we shall show in this note, for the proof itself. But by this simplification we can somewhat improve the lemma and establish the above Lefschetz theorem in the original form.[2)] Further we shall add a remark that the Lefschetz-Hodge's theorem[3)] may be deduced from the fundamental lemma on the same principle.

1. Let V be a compact Kähler variety of complex dim. n, $\mathfrak{U}=\{U_j\}$ a simple covering of V, N its nerve, $\mathfrak{B}$ a complex line bundle defined by $\{f_{jk}\}$ referred to the covering $\mathfrak{U}$, where $f_{jk}(z)$ is a non vanishing holomorphic function in $U_j\frown U_k$ and

$$f_{jk}f_{kj}=1, \qquad f_{jk}f_{ki}f_{ij}=1.$$

We shall redefine the characteristic class $c(\mathfrak{B})$ as follows:[4)]

$$c_{ijk}=h_{jk}+h_{ki}+h_{ij}, \qquad h_{jk}=\frac{\sqrt{-1}}{2\pi}\log f_{jk}. \tag{1}$$

Then, if $\mathfrak{B}$ is defined by the divisor D whose local equation is $f_j(z)=0$ and $f_j/f_k=f_{jk}$, considering D as a current we obtain easily a Weil's chain of double cochains[5)] (coélément)

$$D\longleftrightarrow\Big(\frac{1}{2\pi\sqrt{-1}}d\log f_j\Big)\longleftrightarrow\Big(\frac{\sqrt{-1}}{2\pi}\log f_{jk}\Big)\longleftrightarrow(c_{ijk}),$$

thus D is cohomologous to $c(\{D\})$.[6)]