55. A Note on the Structure of Commutative Semigroups

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The object of the present note is to develop the structure theory of commutative semigroups. By a semigroup we shall always mean a commutative semigroup with identity element 1 and zero element 0.[1] If semigroup S has no identity and zero elements, it can always be imbedded in another S', which has them. S' consists of the elements of S together with new elements 1 and 0. The product of two elements $x, y \in S'$ is defined to be the old product xy of S if $x, y \in S$, otherwise $x0=0=0x$ and $x1=x=1x$ for all $x \in S'$. Moreover, every ideal[2] of S is again an ideal of S' and every principal ideal[3] of S which is generated by an element $x \in S$ is also a principal ideal of S' generated by the same element. Therefore, the assumption that a semigroup has identity and zero elements does not restrict us.

Let S be a semigroup (we recall our convention that "semigroup" means a commutative semigroup with identity element and zero element) and p an element of S, and we define the following (p)-equivalence relation in S:

Two elements a and b of S are *(p)-equivalent* (denoted by $a\overset{p}{\sim}b$) if and only if

$$\bigcap_{n=1}^{\infty}(Sp\cdot a^n)=\bigcap_{n=1}^{\infty}(Sp\cdot b^n).$$

Then it is clear that the (p)-equivalence relation satisfies the following equivalence relations:

(1′) $\quad a\overset{p}{\sim}a$ for all $a \in S$,

(2′) $\quad$ if $a\overset{p}{\sim}b$ then $b\overset{p}{\sim}a$,

(3′) $\quad$ if $a\overset{p}{\sim}b$ and $b\overset{p}{\sim}c$ then $a\overset{p}{\sim}c$.

Now we define the new equivalence relation (denoted by $\sim$), using the above (p)-equivalence relation, in S as follows:

$$a\sim b \textit{ if and only if } a\overset{p}{\sim}b \textit{ for all } p \in S.$$

It is easy to see that the relation $\sim$ satisfies the following equivalence relations:

(1) $\quad a\sim a$ for all $a \in S$,

(2) $\quad$ if $a\sim b$ then $b\sim a$,

(3) $\quad$ if $a\sim b$ and $b\sim c$ then $a\sim c$.

In the discussion below, we denote by S_x the set of all elements in