54. Note on Dirichlet Series. XIII. On the Analogy between Singularities and Order-Directions. II

By Chuji TANAKA

Mathematical Institute, Waseda University, Tokyo

(Comm. by Z. SUETUNA, M.J.A., April 12, 1954)

(1) **Introduction.** Let us put

(1.1) $F(s)=\sum_{n=1}^{\infty} a_n \exp(-\lambda_n s) \quad (s=\sigma+it,\ 0\leqq\lambda_1<\lambda_2<\cdots<\lambda_n\to+\infty).$

In the previous note (1), we have proved

Theorem I (C. Tanaka). *Let (1.1) be uniformly convergent in the whole plane. If we have*

(1.2)
(i) $\Re(a_n)\geqq 0 \quad (n=1, 2, \ldots)$
(ii) $\lim_{n\to\infty} 1/\lambda_n \log\lambda_n . \log(\cos(\theta_n))=0, \quad \theta_n=\arg(a_n),$

then $\Im(s)=0$ *is the order-direction of (1.1).*

In this note, we shall generalize it as follows:

Theorem II. *Let (1.1) be uniformly convergent in the whole plane. Then there exists at least one order-direction in* $|\Im(s)|\leqq\pi\delta$, *provided that*

(1.3)
(i) $\lim_{n\to\infty} 1/\lambda_n \log\lambda_n . \log|\cos(\theta_n)|=0, \quad \theta_n=\arg(a_n),$
(ii) *the sequence* $\{\Re(a_n)\}$ *has sign-changes between* $\Re(a_{p_\nu})$ *and* $\Re(a_{1+p_\nu})$ $(\nu=1, 2, \ldots)$, *where* $\underline{\lim}_{\nu\to\infty}(\lambda_{1+p_\nu}-\lambda_{p_\nu})=g>0$, $\overline{\lim}_{\nu\to\infty} \nu/r_\nu=\delta\ (\leqq 1/g)$, $r_\nu=1/2.(\lambda_{p_\nu}+\lambda_{1+p_\nu})$.

Theorem III. *Let (1.1) be uniformly convergent in the whole plane. Let the subsequence* $\{\lambda_{n_k}\}$ *of* $\{\lambda_n\}$ *be defined as follows:*

(1.4)
(a) $\underline{\lim}_{k\to\infty}(\lambda_{n_{k+1}}-\lambda_{n_k})>0, \quad \underline{\lim}_{\substack{n,k\to\infty\\ n\neq n_k}}|\lambda_n-\lambda_{n_k}|>0,$
(b) $\overline{\lim}_{k\to\infty} k/\lambda_{n_k}=\delta.$

If we have

(1.5)
(i) $\Re(a_n)\geqq 0$, for $n\,\overline{\in}\,\{n_k\}$,
(ii) $\lim_{n\to\infty\ n\overline{\in}\{n_k\}} 1/\lambda_n \log\lambda_n . \log(\cos(\theta_n))=0,$

then in $|\Im(s)|\leqq 2\pi\delta$, *there exists at least one order-direction of (1.1).*

From theorem III follows immediately

Corollary. *Let (1.1) with* $\underline{\lim}_{n\to\infty}(\lambda_{n+1}-\lambda_n)>0$ *be simply (necessarily absolutely) convergent in the whole plane. If we have* $|\arg(a_n)|\leqq\theta<\pi/2$, *except for* $\{a_{n_k}\}$ *such that* $\lim_{k\to\infty} k/\lambda_{n_k}=0$, *then* $\Im(s)=0$ *is the order-direction of (1.1).*