76. Transgression and the Invariant k_n^{q+1}

By Minoru NAKAOKA

Osaka City University, Osaka

(Comm. by K. KUNUGI, M.J.A., May 13, 1954)

§ 1. Let X be a topological space with vanishing homotopy groups $\pi_i(X)$ for $i \neq n, q (1<n<q)$, and let $x_0 \in X$ be a base point. For the sake of brevity, we write in the following $\pi_n = \pi_n(X)$ and $\pi_q = \pi_q(X)$. We call a space of type (π, r) any space Y such that $\pi_i(Y)=0 (i \neq r)$ and $\pi_r(Y) \approx \pi$. Then, following Cartan-Serre,[1] we have the fiber space (E, p, B) in the sense of Serre[2] such that

i) the total space E is of the same homotopy type as X,

ii) the base space B is a space of type (π_n, n), and $X \subset B$,

iii) the fiber $F = p^{-1}(x_0)$ is a space of type (π_q, q).

Consider in this fiber space the transgression τ: $E_{q+1}^{*0,q} \xrightarrow{d^{q+1}} E_{q+1}^{*q+1,0}$ of the singular cohomology spectral sequence with coefficients in π_q.[2] Then, since the singular homology group $H_i(F; \pi_q)=0$ for $i<q$, we have $E_{q+1}^{*0,q} = H^q(F; \pi_q)$, $E_{q+1}^{*q+1,0} = H^{q+1}(B; \pi_q)$ and

$$\tau = p^{*-1} \circ \delta^* : H^q(F; \pi_q) \longrightarrow H^{q+1}(B; \pi_q),$$

where $\delta^* : H^q(F; \pi_q) \longrightarrow H^{q+1}(E, F; \pi_q)$ is the coboundary operator, and $p^* : H^{q+1}(B; \pi_q) \longrightarrow H^{q+1}(E, F; \pi_q)$ is the homomorphism induced by p. Let $\boldsymbol{b}^q \in H^q(F; \pi_q)$ be the basic cohomology class,[3] and let $\overline{\boldsymbol{k}}_n^{q+1} \in H^{q+1}(B; \pi_q)$ be the geometrical realization of the Eilenberg-MacLane invariant $\boldsymbol{k}_n^{q+1} \in H^{q+1}(\pi_n, n; \pi_q)$ of the space X.[4] Then $\boldsymbol{b}^q$ and $\overline{\boldsymbol{k}}_n^{q+1}$ are related by τ as follows:

$$\tau \boldsymbol{b}^q = -\overline{\boldsymbol{k}}_n^{q+1}. \tag{1.1}$$

The main purpose of the present note is to give a proof of (1.1). The proof is given by making use of the theory of J. H. C. Whitehead.[5] In the proof we shall obtain several relations among the various invariants of E, X, B and F. In conclusion, we shall formally extend (1.1) to a more general situation.

§ 2. Following J. H. C. Whitehead,[5] we have the exact sequence $\sum_*(K)$ and the partial exact sequence $\sum^*(K; G)$ for any simply connected CW-complex K and any Abelian group G:

$$\textstyle\sum_*(K) : \cdots \xrightarrow{j_*} H_{r+1}(K) \xrightarrow{d_*} \Gamma_r(K) \xrightarrow{i_*} \Pi_r(K) \xrightarrow{j_*} \cdots,$$

$$\textstyle\sum^*(K; G) : \cdots \xrightarrow{j^*} \Gamma^r(K; G) \xrightarrow{i^*} \Pi^r(K; G) \xrightarrow{d^*} H^{r+1}(K; G) \xrightarrow{j^*} \cdots$$

These are derived from the sequence