# 90. On the Class $S_\lambda$

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§1. **Introduction.** The function $f(x,y)$, which is defined and non-negative in a planer region $D$, is called to belong to the class $S_\lambda$, if the following conditions are satisfied:

(a) $f(x,y)$ is twice continuously differentiable in $D$ and for any point $(x,y)$ in $D$

$$\lim_{r\to 0}\frac{8}{r^2}\left[\left\{\frac{1}{2\pi}\int_0^{2\pi} f(x+r\cos\theta,\ y+r\sin\theta)\,d\theta\right\}^\lambda - \frac{1}{\pi r^2}\iint_{\xi^2+\eta^2\leqq r^2} f^\lambda(x+\xi,\ y+\eta)\,d\xi\,d\eta\right]\geqq 0\,(\lambda>0),$$

or more generally

(b) $f(x,y)$ is the limit of a decreasing sequence $\{f_n(x,y)\}$ each of which satisfies the condition (a).

In particular, when $\lambda=2$, $S_\lambda$ is identical with the class of non-negative subharmonic functions, and when $\lambda=2$, $S_\lambda$ becomes the P.L. class.

The following two theorems for subharmonic function are well-known. The former was proved by T. Radó [1], and the latter by E. F. Beckenbach [2].

**Theorem A.** *If $f(x,y)$ is non-negative in $D$ and if for any pair of two real constants $\alpha$ and $\beta$ the function $\{(x-\alpha)^2+(y-\beta)^2\}f(x,y)$ is subharmonic in $D$, then $f(x,y)$ is a function of the P.L. class in $D$.*

**Theorem B.** *If $f(x,y)$ is non-negative in $D$ and if for any pair of two real constants $\alpha$ and $\beta$ the function $e^{\alpha x+\beta y}f(x,y)$ is subharmonic in $D$, then $f(x,y)$ is a function of the P.L. class in $D$.*

In this paper we shall generalize these theorems to the $S_\lambda$ class.

§2. We require a lemma which plays the fundamental rôle in §3.

**Lemma.** *Let $f(x,y)$ be non-negative, and twice continuously differentiable in $D$.*

*In order that $f(x,y)$ belongs to the class $S_\lambda$, it is necessary and sufficient that*

$$f\Delta f-(\lambda-1)(f_x^2+f_y^2)\geqq 0 \text{ in } D.$$

**Proof.** Let $(x,y)$ be any point in $D$. Without loss of generality we can assume that $f(x,y)>0$ in $D$. Since $f(x,y)$ is twice continuously differentiable in $D$, we have for sufficiently small $r>0$,