112. On the Mass Distribution Generated by a Function of P. L. Class

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§1. **Introduction.** Let $f(x, y)$ be a subharmonic function in a planar region G, and $\mu(e)$ be the completely additive, non-negative Borel set function generated by $f(x, y)$. Let $c(x, y; r)$ be the circle of radius r with center (x, y) included in the region G with its boundary.

We shall introduce the functions:

$$A(f; x, y; r)=\frac{1}{\pi r^2}\int_0^{2\pi}\int_0^r f(x+\rho\cos\theta, y+\rho\sin\theta)\rho d\rho d\theta,$$

$$I(f; x, y; r)=\frac{1}{2\pi}\int_0^{2\pi} f(x+r\cos\theta, y+r\sin\theta)d\theta.$$

Saks[1] proved the following important theorem:

Theorem A. *If $f(x, y)$ is subharmonic in the region G, then, for almost all points (x, y) in G, we have*

$$\lim_{r\to 0}\frac{8}{r^2}[A(f; x, y; r)-f(x, y)]=D_s\mu(x, y),$$

$$\lim_{r\to 0}\frac{4}{r^2}[I(f; x, y; r)-f(x, y)]=D_s\mu(x, y),$$

where $D_s\mu(x, y)$ denotes the symmetric derivative of $\mu(e)$ at (x, y), that is to say,

$$D_s\mu(x, y)=\lim_{\rho\to 0}\frac{\mu[C(x, y; \rho)]}{\pi\rho^2},$$

$C(x, y; \rho)$ being the circle completely included in G.
Recently M. D. Reade[2] proved the following

Theorem B. *If $f(x, y)$ is a function of P. L. class in G, then, for almost all points (x, y) in G, we have*

$$\lim_{r\to 0}\frac{4}{r^2}[I^2(f; x, y; r)-A(f^2; x, y; r)]=f^2(x, y)D_s\sigma(x, y),$$

where $\sigma(e)$ denotes the mass distribution generated by $\log f(x, y)$.
In this paper, we shall generalize this. We shall prove in §2 some lemmas and in §3 our main theorem.

§2. We prove some lemmas which will be used in §3.