111. *Uniform Convergence of Fourier Series*

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J. P. Nash[1] has proved the following theorem.

Theorem 1. *If $f(x)$ is of class $\phi(n)$ with bounded $\phi'(n)$ and is continuous with modulus of continuity $\omega(\delta)$, then there exist positive constants A, B and C independent of $f(x)$ such that*

$$|s_n(x)-f(x)| \leqq \omega\left(\frac{1}{n}\right)\left[A\log\theta(n)+B\frac{n}{\phi(n)}\right]+\frac{C}{\theta(n)},$$

where $\theta(n)$ is monotone increasing and

$$1 \leqq \theta(n) \leqq \phi(n); \quad 1 \leqq \frac{\theta(n+1)}{\theta(n)} \leqq \frac{\phi(n+1)}{\phi(n)}.$$

In this theorem, a function $f(x)$ is said to be of class $\phi(n)$ if

$$\phi(n)\int_a^b f(x+t)\cos nt\,dt = O(1)$$

uniformly for all x, n, a, b with $b-a \leqq 2\pi$.

We shall prove the following generalization which contains the Dini-Lipschitz test as a particular case.

Theorem 2. *If $f(x)$ is of class $\phi(n)$, $\phi(n)$ being $O(n)$,[2] and is continuous with modulus of continuity $\omega(\delta)$, then there exist positive constants A, B and C independent of $f(x)$ such that*

$$(1) \qquad |s_n(x)-f(x)| \leqq \omega\left(\frac{1}{n}\right)\left[A\log\theta(n)+B\log\frac{n}{\phi(n)}\right]+\frac{C}{\theta(n)},$$

where $\theta(n)$ is monotone increasing and $1 \leqq \theta(n) \leqq \phi(n)$.

Proof. It is sufficient to prove (1) for

$$s_n^*(x)-f(x)=\frac{1}{\pi}\int_0^\pi [f(x+t)+f(x-t)-2f(x)]\frac{\sin nt}{2\tan t/2}dt.$$

We divide the integral into three parts such that

$$s_n^*(x)-f(x)=\frac{1}{\pi}\left[\int_0^{\alpha/\phi(n)}+\int_{\alpha/\phi(n)}^{\beta\theta(n)/\phi(n)}+\int_{\beta\theta(n)/\phi(n)}^{\pi}\right]$$
$$=\frac{1}{\pi}[I+J+K]$$

1) J. P. Nash: Uniform convergence of Fourier series, The Rice Institute Pamphlet (1953).

In this paper we use the notation in Zygmund, Trigonometrical series, 1936.

2) As J. P. Nash shows, the assumption $\phi(n)=O(n)$ does not loose generality.