152. On Isometric Analytic Functions in Abstract Spaces

By Isae SHIMODA
Mathematical Institute, Gakugei College, Tokushima University

(Comm. by K. KUNUGI, M.J.A., Oct. 12, 1954)

In classical analysis, if a complex valued function $f(z)$ is regular and satisfies $|f(z)|=|z|$ in a circle $|z|<R$ in complex plane, then $f(z)=e^{i\theta}z$, that is, $f(z)$ is linear. We expect that an isometric analytic function is also linear in complex Banach spaces. An isometric analytic function is linear in fact if it is analytic on whole spaces but is not necessarily linear if it is defined on a bounded domain in complex Banach spaces.

Let E, E' be complex Banach spaces and α, β be complex numbers. An E'-valued function defined on a domain D in E is *analytic*[1,2] in D, if it is continuous strongly and admits Gateaux differential in D. An analytic function $h_n(x)$ which satisfies $h_n(\alpha x)=\alpha^n h_n(x)$ for an arbitrary complex number α and an arbitrary point x in E is called *a homogeneous polynomial of degree* n.[3]

Theorem 1. If an E'-valued function $f(x)$ is analytic in the sphere $\|x\|<R$ in E and satisfies $\|f(x)\|\leq K\|x\|^m$, then

$$f(x)=\sum_{n=m}^{\infty}h_n(x),$$

where R, K and m are constants and $h_n(x)$ is a homogeneous polynomial of degree n.

Proof. Since $f(x)$ is analytic in the sphere $\|x\|<R$, we have

$$f(x)=\sum_{n=0}^{\infty}h_n(x),$$

where $h_n(x)$ is a homogeneous polynomial of degree n and

$$h_n(x)=\frac{1}{2\pi i}\int_C\frac{f(\alpha x)}{\alpha^{n+1}}\,d\alpha,$$

for $n=1, 2, 3, \dots$ and C is a circle $|\alpha|=r$ such that $1<r<\dfrac{R}{\|x\|}$. Then, we have

$$\|h_n(x)\|\leq\left\|\frac{1}{2\pi i}\int_C\frac{f(\alpha x)}{\alpha^{n+1}}\,d\alpha\right\|\leq\frac{1}{2\pi}\int_0^{2\pi}\frac{\|f(re^{i\theta}x)\|}{r^n}\,d\theta$$
$$\leq\frac{1}{2\pi}\int_0^{2\pi}\frac{K\|re^{i\theta}x\|^m}{r^n}\,d\theta=Kr^{m-n}\|x\|^m,$$

since $\|f(\alpha x)\|\leq K\|\alpha x\|^m$ for $\|\alpha x\|<R$. Since r can be taken as close as we like to 1, we have $\|h_n(x)\|\leq K\|x\|^m$, for $\|x\|<R$. Clearly, $\|\beta x\|<R$ if $|\beta|<1$, and we have $\|h_n(\beta x)\|\leq K\|\beta x\|^m$. Then, $|\beta|^n\|h_n(x)\|\leq K|\beta|^m\|x\|^m$. If $n<m$, $\|h_n(x)\|\leq K|\beta|^{m-n}\|x\|^m$, for $0<|\beta|\leq 1$. Let β