145. On the Characterization of the Harmonic Functions

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§1. **Introduction.** The well-known Green formula for functions of two variables, may be stated as follows:

$$(1)\quad \iint_R \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\,dxdy+\iint_R v\Big(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\Big)dxdy=\int_C v\frac{\partial u}{\partial n}\,ds,$$

where $u(x, y)$ and $v(x, y)$ are functions of class C^2 and R is a bounded planar region with boundary C. Then, from (1) we have

Theorem 1. *If u and v are harmonic in R, then*

$$(2)\quad 2\iint_R \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\,dxdy-\int_C\Big(v\frac{\partial u}{\partial n}+u\frac{\partial v}{\partial n}\Big)ds=0.$$

In §2, we shall prove a theorem which is a sort of inverse of Theorem I. For the proof, we use the method due to Beckenbach [1]. On the other hand it is known that

Theorem 2. *If $u(x, y)$ is harmonic in a planar domain R, then for any closed circle $C(x, y; r)$ contained in R.*

$$(3)\quad \frac{1}{2\pi}\int_0^{2\pi} u(x+r\cos\theta,\ y+r\sin\theta)\,d\theta - \frac{1}{\pi r^2}\int_0^{2\pi}\int_0^{r} u(x+\rho\cos\theta,\ y+\rho\sin\theta)\,\rho d\rho d\theta=0.$$

Further Levi [2] and Tonelli [3] proved that if $u(x, y)$ is continuous in R and (3) holds for any closed circle C contained in R, then $u(x, y)$ is harmonic in R.

We prove a similar theorem in §3.

§2. **Lemma 1** (Saks [4]). *If $u(x, y)$ belongs to the class C^1 and for any closed circle $C(x, y; r)$ contained in D*

$$\int_0^{2\pi}\frac{\partial u}{\partial n}\,rd\,\theta=o(r^2),^{1)}$$

then, $u(x, y)$ is harmonic in D.

As an inverse of Theorem 1, we prove

Theorem I. *If $u(x, y)$ and $v(x, y)$ belong to the class C^1 in a*

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1) $\phi(r)=o(r^\alpha)$ means that $\lim\limits_{r\to 0}\dfrac{\phi(r)}{r^\alpha}=0$.