177. A Characterization of Hilbert Space

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It is our purpose in this note to prove the following

THEOREM. *A Banach space E is unitary if and only if it satisfies the condition.*

(*) *There is assigned to E a positive number α not greater than 1/2, and for any x, y in E, there exists at least a λ, $\alpha \leqq \lambda \leqq 1-\alpha$, which depends on x and y, such that*

$$\lambda \| x \|^2 + (1-\lambda) \| y \|^2 \geqq \lambda(1-\lambda) \| x-y \|^2 + \| \lambda x + (1-\lambda) y \|^2,$$

where $\| \ \|$ is the norm.

Whenever we speak of a Banach space we shall mean a Banach space over real field R.

We shall only prove the "if" part of the theorem since the "only if" part is clear. Using Kakutani's result,[1] it is sufficient to show that for any closed linear subspace M of E, there exists an extension of the identity transformation of M which is linear continuous and has norm 1. From the fact that the continuous linear map of a linear subspace N of a Banach space into another Banach space F can be extended to a continuous linear map of the closure $\bar{N}$ into F without changing the norm, and by virtue of Zorn's lemma, our problem can be simplified in the form: to prove the following statement.

Let E be a Banach space satisfying the condition (*), and M a closed hyperplane. Then the identity transformation I of M can be extended to a continuous linear transformation of E onto M whose norm is 1.

For this purpose, we shall need the lemmata below.

LEMMA 1. *Let E be a Banach space satisfying the condition (*). If x, $y \in E$ are such that:*

$$\max [\| x \|, \ \| y \|] < \| x-y \|$$

then there is a λ, $0 < \lambda \leqq 1-\alpha$, which insures

$$\| \lambda x + (1-\lambda) y \| < \min [\| x \|, \ \| y \|].$$

Proof. We may suppose $\| y \|$ is not greater than $\| x \|$. In 2-dimensional Euclidean space, we constract a triangle with verteces

1) S. Kakutani: *Some characterizations of Euclidean space*, Japanese Jour. Math., **16**, 93–97 (1939).