175. On Completely Continuous Operators on Locally Convex Vector Spaces

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The classical theorem of Riesz and Schauder concerning completely continuous operators on Banach spaces was extended on locally convex vector spaces by J. Leray.[1] However the proof of the alternative of Fredholm was based on a theorem on the invariance of the domain and the theorems of Schauder for the conjugate space were incomplete.

In the present note we prove the alternative of Fredholm and the Riesz-Schauder theory in the case of locally convex vector spaces. Furthermore we concern ourselves with iterations of continuous operators and with results of M. Altman.[2] The method is linear space-theoretic and is based on the Leray's results.

1. The alternative of Fredholm. Let X be a locally convex vector space with complex coefficients and let t be a completely continuous operator on X into itself, i.e., there is a neighbourhood U of 0 in X such that $t(U)$ is relatively compact (compact=bicompact). Furthermore let X^* be the space of all continuous functionals on X with the compact open topology. Then we have the following

Lemma 1. The adjoint t^* of t is completely continuous with respect to the above topology on X.

For $t^*((t(U))^0)\subset U^0$ where $U^0=\{f\mid f(U)\mid\leqq 1\}$ and U^0 is compact in our topology by the theorem of Ascoli.

Lemma 2. Let t be a completely continuous operator without eigenvalue 1. Then $(t-1)(X)=X$.

For let $v=t-1$ and suppose, on the contrary, that $v(X)\subsetneqq X$. Then by Lemma 1, t^* is completely continuous and $v^{*^{-1}}(0)=(v(X))^0$, hence by [L], Lemma 9.2, $v^{*^{-\rho}}(0)=v^{*^{-\rho-1}}(0)\neq\{0\}$ for some integer ρ. Accordingly $v^\rho(X)=v^{\rho+1}(X)$. For $x\in v^\rho(X)-v^{\rho+1}(X)$ implies that

1) Cf. J. Leray: Valeurs propres et vecteurs propres d'un endomorphisme complètement continu d'un espace vectoriel à voisinages convexes, Acta Scient Math., **12** B (1950), which will be referred to as [L].

2) When I have almost finished the present note, Professor M. Nagumo called my attension to M. Altman's results: M. Altman: On linear functional equations in locally convex linear topological spaces, Studia Math., **13** (1953).