172. The Divergence of Interpolations. II

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Next we shall consider the function analytic interior to the circle C_R and with singularities of Y_m type on C_R. Such functions can be constructed by

$$f(z)=\varphi(z)+\sum_{k=1}^{N}\varphi_k(z)y_{m_k}(z;a_k);\quad a_k=Re^{i\alpha_k}, \tag{13}$$

where $\varphi(z)$ and $\varphi_k(z)$ are functions single valued and analytic on and within the circle C_R, and a_k are points on C_R not necessarily distinct. For such functions, we have the following theorem.

Theorem 2. *Let $P_n(z;f)$ be partial sums of the power series of $f(z)$ represented by (13). Then*

$$\overline{lim}_{n\to\infty}\left|n^p\left(\frac{R}{z}\right)^n P_n(z;f)\right|>0 \quad \text{for} \quad |z|>R, \tag{14}$$

where p is the minimal real part of m_k in (13). Accordingly, $P_n(z;f)$ diverges at every point exterior to the circle C_R as n tends to infinity.

In the proof of this theorem, it is convenient to have the following lemma.

Lemma 3. *Let A_k; $k=1, 2, \dots, N$ be a given set of complex numbers not all equal to zeros. Let α_k; $k=1, 2, \dots, N$ be mutually distinct angles between zero and 2π, and q_k; $k=1, 2, \dots, N$ be a set of real numbers. Then we have*

$$\overline{lim}_{n\to\infty}\left|\sum_{k=1}^{N}A_k e^{-i(q_k\log n+n\alpha_k)}\right|>0. \tag{15}$$

For a real number q not equal to zero, the relation

$$e^{-i(q\log n+n\alpha)}=\frac{1}{\Gamma(iq)}\int_0^\infty e^{-n(t+i\alpha)}t^{iq-1}dt;\ n=1, 2, \dots$$

can be verified by the well-known formula

$$\Gamma(iq)=\int_0^\infty e^{-x}x^{iq-1}dx.$$

Then we have for α not equal to zero

$$\frac{1}{n}\sum_{\nu=1}^{n}e^{-i(q\log\nu+\nu\alpha)}=\frac{1}{n\Gamma(iq)}\int_0^\infty\frac{1-e^{-n(t+i\alpha)}}{1-e^{-(t+i\alpha)}}e^{-(t+i\alpha)}t^{iq-1}dt$$

$$=\frac{1}{\Gamma(iq+1)}\left\{\frac{1}{n}\int_0^\infty\frac{e^{-2(t+i\alpha)}(1-e^{-n(t+i\alpha)})}{[1-e^{-(t+i\alpha)}]^2}t^{iq}dt\right.$$

$$\left.-\int_0^\infty\frac{e^{-(n+1)(t-i\alpha)}}{1-e^{-(t+i\alpha)}}t^{iq}dt+\frac{1}{n}\int_0^\infty\frac{e^{-(t+i\alpha)}(1-e^{-n(t+i\alpha)})}{1-e^{-(t+i\alpha)}}t^{iq}dt\right\}$$