167. A Note on Strongly (C, α)-ergodic Semi-Group of Operators

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Let $\{T(\xi): 0<\xi<\infty\}$ be a semi-group of operators satisfying the following assumptions:

(i) *For each* ξ, $0<\xi<\infty$, $T(\xi)$ *is a bounded linear operator from a complex Banach space* X *into itself and*

$$(1)\qquad T(\xi+\eta)=T(\xi)T(\eta).$$

(ii) $T(\xi)$ *is strongly measurable in* $(0, \infty)$.

$$\text{(iii)}\qquad \int_0^1 \|T(\xi)x\|\,d\xi<\infty \qquad \textit{for each } x\in X.$$

We may further assume without loss of generality that

(iv) $\|T(\xi)\|$ *is bounded at* $\xi=\infty$.

If $T(\xi)$ satisfies the condition

$$\text{(v)}\qquad \lim_{\lambda\to\infty}\lambda\int_0^\infty e^{-\lambda\xi}T(\xi)x\,d\xi=x \qquad \textit{for each } x\in X,$$

then $T(\xi)$ is said to be strongly *Abel-ergodic* to the identity at zero. If, instead of (v), $T(\xi)$ satisfies the stronger condition

$$\text{(v}'\text{)}\qquad \lim_{\xi\to 0}\alpha\xi^{-\alpha}\int_0^\xi(\xi-\eta)^{\alpha-1}T(\eta)x\,d\eta=x \qquad \textit{for each } x\in X,$$

then $T(\xi)$ is said to be strongly *(C, α)-ergodic* to the identity at zero.

Recently R. S. Phillips [1] and the present author [3] have independently proved the following

Theorem 1. *A necessary and sufficient condition that a semi-group of operators strongly Abel-ergodic to the identity at zero be of operators strongly* $(C, 1)$*-ergodic to the identity at zero is that there exists a positive number* M *such that*

$$(2)\qquad \sup_{k\geq 1,\lambda>0}\left\|\frac{1}{k}\sum_{i=1}^{k}[\lambda R(\lambda; A)]^i\right\|\leq M.$$

In this note we shall give a generalization of Theorem 1 which is stated as follows:

Theorem 2. *Let* α *be a positive integer. A necessary and sufficient condition that a semi-group of operators strongly Abel-ergodic to the identity at zero be of operators strongly* (C, α)*-ergodic to the identity at zero is that there exists a positive number* M *such that*

$$(3)\qquad \sup_{\lambda>0,k\geq\alpha}\left\|\frac{\alpha}{k(k-1)\cdots(k-\alpha+1)}\sum_{i=1}^{k-\alpha+1}\frac{(k-i)!}{(k-\alpha+1-i)!}[\lambda R(\lambda; A)]^i\right\|\leq M.$$