201. *Harmonic Measures and Capacity of Sets of the Ideal Boundary. I*

By Zenjiro KURAMOCHI
Mathematical Institute, Osaka University
(Comm. by K. KUNUGI, M.J.A., Dec. 13, 1954)

Let R be an abstract Riemann surface of positive boundary and let $\{R_n\}$ $(n=0,1,2,\dots)$ be its exhaustion with compact relative boundaries $\{\partial R_n\}$.[1] Each ∂R_n consists of a finite number of analytic curves. Let D be a non compact subdomain whose relative boundary ∂D consists of at most an enumerably infinite number of analytic curves clustering nowhere in R. We say that a sequence $\{D\cap(R-R_n)\}$ determines a subset of the ideal boundary, which is denoted by B_D. In this article we shall introduce the harmonic measures and capacity of B_D and study their applications.

1. Harmonic Measures

Let $U(z)$ be a continuous function in R. If there exists a number n such that $U(z)\geqq 1-\varepsilon$ for given ε in $D\cap(R-R_n)$, we say that $U(z)$ has limit $\geqq 1$ in B_D. Let $\omega_{n,n+i}(z)$ be a bounded harmonic function in $R_{n+i}-((R_{n+i}-R_n)\cap D)$ such that $\omega_{n,n+i}(z)=0$ on $\partial R_{n+i}-D$ and $\omega_{n,n+i}(z)=1$ on $(\partial R_n\cap D)+(\partial D\cap R_{n+i})$. Then $\omega_{n,n+i+j}(z)\geqq\omega_{n,n+i}(z)$ and $\omega_{n+i,j}(z)\leqq\omega_{n,j}(z)$. Put $\lim\limits_{n=\infty}\lim\limits_{i=\infty}\omega_{n,n+i}(z)=\omega(z)$. We call $\omega(z)$ the outer harmonic measure of B_D. We define the inner harmonic measure of B_D similarly. Another definition is as follows: Let $\{v(z)\}$ be a class of continuous super-harmonic functions such that $0\leqq v(z)\leqq 1$, $\lim v(z)\geqq 1$ in B_D. Let $V(z)$ be its lower envelope. Then it is easy to prove that $V(z)=\omega(z)$. Let R_0 be a compact disc in R and let $\omega'_{n,n+i}(z)$ be a bounded harmonic function in $R_{n+i}-((R_{n+i}-R_n)\cap D)-R_0$ such that $\omega'_{n,n+i}(z)=0$ on $\partial R_0+(\partial R_{n+i}-D)$ and $\omega'_{n,n+i}(z)=1$ on $(\partial R_n\cap D)+(\partial D\cap R_{n+i})$.

Then $\lim\limits_{n=\infty}\lim\limits_{i=\infty}\omega'_{n,n+i}(z)=\omega'(z)$. We have at once from the definition the follwing

Theorem 1. *Let B_{D_1} and B_{D_2} be two subsets of ideal boundary and let $\omega_{D_i}(z)$ be harmonic measures of B_{D_i}. Then*

$$\omega_{D_1}(z)+\omega_{D_2}(z)\geqq\omega_{D_1+D_2}(z),\quad \omega'_{D_1}(z)+\omega'_{D_2}(z)\geqq\omega'_{D_1+D_2}(z).$$

If $D'\supset((R-R_m)\cap D)$ for a number m, we say that D' covers B_D. Let $D_1\supset D_2,\dots$ be a sequence of non compact domains containing B_D and let $U(z)$ be a positive harmonic function in R. We denote

1) In this article, we denote by ∂G the relative boundary of G.