195. A Simple Proof of Littlewood's Tauberian Theorem

By Shin-ichi IZUMI

(Comm. by Z. SUETUNA, M.J.A., Dec. 13, 1954)

Littlewood's tauberian theorem[1] reads as follows:

If $\sum_{n=0}^{\infty} a_n x^n$ *converges for* $|x|<1$ *and*

$$\lim_{x\to 1}\sum_{n=0}^{\infty} a_n x^n = 0, \tag{1}$$

$$|a_n| \leqq A/n \qquad (n=1, 2, \dots), \tag{2}$$

then $\sum_{n=0}^{\infty} a_n = 0$.

Two simple proofs of this theorem were given by J. Karamata.[2] We shall give another simple one.

Let

$$a(t)=a_n \qquad (n \leqq t < n+1,\ n=1, 2, \dots),$$

$$S(t)=\int_0^t a(u)du,$$

then[3]

$$f(s)=\int_0^\infty a(t)e^{-st}dt=\sum_{n=0}^{\infty} a_n e^{-st}\int_0^1 e^{-st}dt.$$

Hence the conditions (1) and (2) become

$$|a(t)| \leqq A/t, \quad \lim_{s\to 0} f(s)=0. \tag{3}$$

We shall further put

$$P_u(t)=e^{-ut}\sum_{m<u}\frac{(ut)^m}{m!},$$

$$g(t)=1 \ (0\leqq t<1), \qquad g(t)=0 \ (t>1).$$

Then we have

$$\begin{aligned} S(T) &= \int_0^T a(t)dt = T\int_0^1 a(Tt)dt = T\int_0^\infty a(Tt)g(t)dt \\ &= T\int_0^\infty a(Tt)[g(t)-P_u(t)]dt + T\int_0^\infty a(Tt)P_u(t)dt \\ &= S_1(T)+S_2(T), \end{aligned}$$

say. By (3)

$$|S_1(T)| \leqq A\int_0^\infty \frac{1}{t}\,|g(t)-P_u(t)|\,dt$$

---

1) J. E. Littlewood: Proc. London Math. Soc., **9** (1918).

2) J. Karamata: Math. Zeits., **32** (1932); **56** (1952). Cf. K. Knopf: Theorie der unendlichen Reihen, 2te Aufl. R. Wielandt has given a simple proof of the one side theorem due to G. H. Hardy and J. E. Littlewood in Math. Zeits., **56** (1952).

3) This is the form used by A. Korevaar in his papers in Indak. Math. (1953–1954).