5. Vector-space Valued Functions on Semi-groups. I

By Kiyoshi ISÉKI

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S. Bochner and J. v. Neumann (1) extended the theory of almost periodic functions to functions whose values are elements of vector spaces. In recent papers, W. Maak (2), (3) discussed almost periodic functions defined on semi-groups. W. Maak theory may be regarded a generalisation of J. v. Neumann theory of almost periodic functions on groups (4). In these Notes, we shall consider vector valued functions on semi-groups.

By a semi-group G, we shall mean an algebraic system in which a multiplication is defined, and the law of composition has the associative property. A locally convex space E is a topological vector space over real field in which there is a fundamental system of neighborhoods (briefly n.b.d.) of O which are convex. A locally convex space E is (F)-space, if it is metrisable and complete. Throughout what follows, all functions considered are to be mappings of a semi-group G into a locally convex space E.*)

I. The definition of almost periodic functions

Definition 1. A function $f(x)$ is called almost periodic, if, for a given n.b.d. U,[1] *there are finite family of subsets $A_1, A_2, \dots, A_n$ of G such that*

(1) $$G=\bigcup_{i=1}^{n} A_i$$

(2) $$c'x'd',\ c'y'd' \in A_i \text{ implies}$$
$$f(cxd)-f(cyd) \in U \text{ for all } c, d \in G.$$

By the decomposition $\{f(x), U\}$ of G, we shall mean the sets $\{A_i\}$ $(i=1, 2, \dots, n)$ in Definition 1. It is clear that any constant function on G is almost periodic.

Theorem 1. Let G be a group and $f(x)$ vector valued function. The function $f(x)$ is almost periodic, if and only if, for any n.b.d. U, there are subsets $A_1, A_2, \dots, A_n$ of G such that

(1)′ $$G=\bigcup_{i=1}^{n} A_i$$

(2)′ *$x, y \in A_i$ implies $f(cxd)-f(cyd) \in U$ for c, d of G.*

Proof. If $f(x)$ is almost periodic, it is clear that a decomposition $\{f(x), U\}$ satisfies the conditions (1)′ and (2)′.

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*) For details of locally convex spaces, see J. Dieudonné: Recent developments in the theory of locally convex vector spaces, Bull. Amer. Math. Soc., **59**, 495–512 (1953).

1) n.b.d. U means convex n.b.d. U.