3. *Notes on the Riemann-Sum*

By Shigeru TAKAHASHI
Department of Mathematics, Kanazawa University, Japan
(Comm. by Z. SUETUNA, M.J.A., Jan. 12, 1955)

§ 1. Let $\{t_i(w)\}$ $i=1, 2, \ldots$ be a sequence of independent random variables in a probability space (Ω, B, P) and each $t_i(w)$ has the uniform distribution in $[0, 1]$, that is

(1.1) $$F(x)=P(t_i(w)<x)$$

which is 1, x, or 0 according as $x>1$, $0\leqq x\leqq 1$ or $x<0$. For each w, let $t_i^{(n)}(w)$ denote the i-th value of $\{t_j(w)\}$ $(1\leqq j\leqq n)$ arranged in the increasing order of magnitude and let

(1.2) $$t_0^{(n)}(w)\equiv 0, \qquad t_{n+1}^{(n)}(w)\equiv 1, \qquad (n=1, 2, \ldots).$$

Further let $f(t)$ $(-\infty<t<+\infty)$ be a Borel-measurable function with period 1 and belong to $L_1(0, 1)$.

Professor Kiyoshi Ito has recently proposed the problem: Does

(1.3) $$S_n(w)=\sum_{i=1}^{n} f(t_i^{(n)}(w))(t_i^{(n)}(w)-t_{i-1}^{(n)}(w))$$

converge to $\int_0^1 f(t)dt$ in any sense?

In this note, we consider the following translated Riemann-sum

(1.4) $$S_n(w, s)=\sum_{i=1}^{n} f(t_i^{(n)}(w)+s)(t_i^{(n)}(w)-t_{i-1}^{(n)}(w))$$

and prove the following

Theorem 1. *Let $f(t)$ be $L_2(0, 1)$-integrable and for any $\varepsilon>0$,*

(1.5) $$\left(\int_0^1 |f(t+h)-f(t)|^2 dt\right)^{1/2}=O\left(1\Big/\left|\log\frac{1}{|h|}\right|^{1+\varepsilon}\right) \qquad (|h|\to 0).$$

Then for any fixed s, we have

$$P\left(\lim_{n\to\infty} S_n(w, s)=\int_0^1 f(t)dt\right)=1.$$

Remark. The w-set on which $S_n(w, s)\to\int_0^1 f(t)dt$ depends on s.

Theorem 2. *Let $f(t)$ be $L_1(0, 1)$-integrable and for an $\varepsilon>0$,*

(1.6) $$\int_0^1 |f(t+h)-f(t)|\, dt=O\left(1\Big/\left|\log\frac{1}{|h|}\right|^{1+\varepsilon}\right) \qquad (|h|\to 0).$$

Then for any fixed w, except a w-set of probability zero, there exists a set $M_w\subset[0, 1]$ with measure 1 such that

$$\lim_{n\to\infty} S_n(w, s)=\int_0^1 f(t)dt \qquad (s \in M_w).$$

§ 2. By (1.1) and the independency of $\{t_i(w)\}$, it may be seen that

(2.1) $$P(\bigcup_{m\neq n}(t_m=t_n))=0.$$